

F.E.D. Vignette #4, Part II. --

The Gödelian Dialectic ***of*** ***the Standard Arithmetics***

A Dialectical 'Meta-Model' of the F.E.D. 'Meta-Systematic Dialectical Method of Presentation'
of the Axioms-Systems Progression of the Standard Arithmetics, Using F.E.D.s 'First Dialectical Algebra'.

‘Habilitation-«Fest»’ Essay

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C. Core: ‘Organonic Method’ Applied to Solve the Dialectical Equation for Our Presentation of the Standard Arithmetics.

This whole progression of the Standard Arithmetics is a ‘self-argument’ -- a series of arguments with, and within, and against each stage of itself; a series of **assertions** of number-meaning, of **objections** to those assertions, and of mutual **reconciliations** of those assertions and of their objections. That is, this whole progression is a **dialectic** of/about number ‘ideo-ontology’, i.e., about numbers’ ‘conceptual ontology’; about the definition of the term ‘Standard Number’.

The «**arché**» arithmetic, $\overset{3}{H}\overset{\#}{N}$, provides the final epithet of each **full-synthesis term**, or ‘**core uni-thesis**’; of each **stage**; of each **system-of-arithmetic epitome**; of each **culminant** --



-- and is always about the **accretions of thereby-‘explicitized’ number-meaning** to that “‘core’”, accretions which are the fruitions of every passing **Step/Stage** of this ‘self-argument’, contributed by every newly arising ‘contra-thesis’ epithet, or “‘determination’”.

Every succeeding ‘contra-thesis’ term in this progression of terms is the result/fruition of the self-subsumption/-“**auto-critique**” of its **predecessor** ‘contra-thesis’ term, criticizing in accord with that **predecessor**’s own, internal standards. Every incremental term preceding such a ‘contra-thesis’ term, in the entire terms-increment of the **stage/Step** in which that ‘contra-thesis’ term arises, as that **Stage**’s/**Step**’s final term, is the fruition of an ‘**allo-critique**’ of an “other”, previously-arisen term, by the **predecessor** ‘contra-thesis’ term, critiquing in accord with the internal standards of that **predecessor** ‘contra-thesis’ term. Each such “other” is the result of either the «**arché**» term, or one of the ‘partial synthesis’ terms [if any], or the preceding **stage**’s/**Step**’s ‘full synthesis’/‘culminant’ term, being critiqued by that **predecessor** ‘contra-thesis’ term.

This progression thus does encompass ‘**allo-critique**’, not just ‘**auto-critique**’ or *immanent* critique, relative to the internal perspective of a particular ‘contra-thesis’, within the ‘cumulum’ that is each **S > 1** **step/Stage** of its ‘self-argument’/‘self-dialogue’. However, from the perspective of whole **stages**, each succeeding **stage** is always the product of *self-critique*, or *immanent* critique, only -- of the “‘squaring [with itself]’” -- of the entire **stage** which immediately preceded it.

In this -- dialectical -- context, “criticism” is not “abstract negation” -- ‘**annihilatory opposition**’: an attack upon its object which obliterates or erases that object from existence, leaving only an abstract nothing in place of the something that had been there before that criticism. Dialectical criticism is “‘determinate negation’”, a negation/modification of [some of] the “determinations” of its object, which also includes a preservation of [some of its [other]] “determinations” as well. Such criticism is negation through “‘conservative extension’”, through past-conserving ‘futurization’ of ‘past-and-present’, including of ‘present-past’, by the ‘self-operation’ of the present. It is a past-[and-]present-preserving -- but also an [ideo-]ontologically innovative -- **supplementation**; ‘**supplementary opposition**’, via the **continuing** [oppositional] “‘addition’” [-to-self] of correctives, of “‘addenda’”, of ‘amendatory appenda’, of “‘coda’”, of “‘codicils’”, including of many that are **major**, and that also do not erase, but that do modify, their predecessors.

This progression is a ‘self-complexifying’, qualitatively/[‘ideo-’]ontologically ‘self-compounding’, compound ‘qualitative-returns’-delivering, self-expanding ‘cumulum’ of ideas. It is a ‘**stagedly**’, self-similarly recurring, but never-exactly-repeating ongoing “‘snowballing’”; a continual layered accretion of ever-new numerical and arithmetical “determinations” [cf. Hegel].

With that general description of the dialectical progression of the “Standard Arithmetics” now behind us, we are ready to go forward, into that progression’s full specifics and details at last!

“‘Person-ification’” and ‘Im-Person-ation’. The physical terms, the tangible ideographical symbols, that the core ‘meta-model’ of this essay sums, are not, in any sense, in themselves, subjects, or agents -- centers that *initiate* action -- in their own right. How could they be? They are but, e.g., small, solidified pools of toner, adhering to paper.

Of course, to those who share in the “‘inter-subjectivity’” for which these desiccated droplets make meaningful marks, those marks evoke, whenever those ‘sharers’ read them, specific meanings, particular ideas. But these ideas live, so far as we know, only inside individual human minds. Ideas are vivified only by living human beings, forming them, holding them in mind. Ideas may have some minimal subconscious, unintentional ‘subject-hood’, some agency, in a human mind, once willfully formed in that mind by action of its ‘mind-er’. But almost all of any ‘agent-hood’, or ‘subject-ness’, that ideas possess, is consciously lent to them by each human subject who forms them in mind, in response to, e.g., human speech, or to some textual symbol(s). Their ‘subject-ivity’ is borrowed from the real subjects. Their ‘agent-ness’ persists only when, and only while, they are being “‘person-ified’” or ‘[im-]person-ated’ -- made into ‘pseudo-persons’ -- by a real person. To believe otherwise is fetishism, that signal symptom of ideology, of the failure of science -- like the “Fetishism of Commodities”, the fetishism of Money, the fetishism of Capital, the fetishism of [exchange-]Value in general -- that Marx so devastatingly diagnosed in the ideology-compromised science of classical capitalist political economy. To believe otherwise would be a ‘fetishism of Ideas’, akin to the ideology to which Plato’s Socrates -- and to which at least the early Plato as well, prior to The Parmenides -- succumbed: not to mention so many others since!

These physical symbols -- these ‘empapered’ patterns of ink, staining the page -- are dead; a deceased residue of past, ended thought-life, that once guided the hand that wrote down their ‘conventioned’ representatives as marks on ...papyrus..., parchment..., paper, as human mind-remains, ‘psychoartefacts’. And dead they remain -- unless a living person enlivens them, by *comprehendingly* reading them, and by [re-]*cognizing* them”: “‘impersonating’” them -- infusing them with living personality, with living human subjectivity, with active agency -- by thinking them, and therefore also by “‘incarnating’” them in that person’s seemingly flesh-less mind; by “‘mentally embodying’” them, in that person’s seemingly ‘body-less’, ‘dis-embodyed’” mind -- as acting, interacting, [self-]critiquing and [self-]changing human thoughts, residing, for a time, within the space of self-aware consciousness of a breathing being.

The conclusion with which we are left is that what these symbols really represent are human acts, human cognitive acts -- “Mental Operations” [cf. Boole]. It is people -- human persons -- who animate the **Cs** and the **Ms** of Marx’s $\mathbf{C} \Leftarrow \mathbf{M} \Leftarrow \mathbf{C}$ ’s and $\mathbf{M} \Leftarrow \mathbf{C} \Leftarrow \mathbf{M}$ ’s, who stand behind, and act behind, who ‘enmask’ themselves with -- who “personify” [Marx] -- these “social relations of production”.

Likewise, it is people who animate that which the arithmetical-system category-operators -- $\overset{3}{\mathbf{H}}\overset{\mathbf{N}}{\#}$, $\overset{3}{\mathbf{H}}\overset{\mathbf{a}}{\#}$, $\overset{3}{\mathbf{H}}\overset{\mathbf{m}}{\#}$, $\overset{3}{\mathbf{H}}\overset{\mathbf{f}}{\#}$, etc., denote. Like each fictional character of a famous novel, made into a movie, the ‘idea-eventities’ which these ‘connotograms’ and ‘categorograms’ conjure in the consciousnesses of their ‘cognizers’ can live and act only if impersonated by a human person, by a human subject, by a human actor. Terran humanity is the real, collective subject, the real agent, of the Gödelian Dialectic of the “Standard Arithmetics”, the real agent of the “self”-critique/“self”-elaboration of the $\overset{3}{\mathbf{H}}\overset{\mathbf{N}}{\#}$, driven by that humanity’s expanding human-society-reproductive activity, that activity of the “‘human phenome’”, and not any ‘phantastic’, merely imagined, supposedly disembodied, ‘dis-en-minded’ “Idea”.

Objects, including even pre-human/extra-human living [ev]entities’, other biological beings, do not ‘self-awarely’ enact dialectical critique. To our knowledge, only humans can enact true critique. Therefore, the dialectical-ideographical symbols employed in this essay, the operator symbols for immanent critique that constitute our main ‘meta-model’, must denote intuitive operations, operations that can only be carried out by human subjects: they denote the operations of human minds. What these symbols symbolize are [trans-Boolean] human mental movements, “‘dialogical’” and ‘self-dialogical’ mental activities of human beings. In the last analysis, the formulae of our ‘meta-model’ evoke a description of, or a guide to, one’s own thought process, one’s own self-dialogue, in process of considering the meaning/definition of number in modern/contemporary Standard Arithmetic(s). These formulae are ‘mind-guides’, ‘replayable’ condensed recordings of, e.g., past, polished, proven-to-be-advantageous ‘thought-trains’/ ‘thought-sequences’ / thought-progressions -- “‘programs’”/ “‘software’”, not for a digital computer, but for a human mind; ‘thought-recipes’ & ‘thought-guides’; scores for symphonies of thought. Our written-out recordings are a means for presently following the past thoughts of others, or of ourselves, thoughts that left behind a “‘fossil record’” in tangible, written form, a form that can be deciphered/‘re-minded’ to ‘re-navigate’ present readers’ thoughts, anew, down mind-roads of old, on trails blazed before, by others.

If you are “following” the categorial progression/systems-progression modeled herein -- conjuring up for yourself, in your own mind, ‘similants’ of the connotations and intuitions of the axioms-systems that its terms interpret -- then its symbols, its formulae, its equations, are, thereby, now *about you*. ¡These symbols are now describing and guiding what is going on in your own mind while you read them, and while you think them! They are now describing, as well as steering, your own thoughts. The symbols of this progression of symbols are symbolizing the progression of your own thoughts now. All of this algebra is describing your own mental operations now. All of this ideography is “‘graphing’” the flow of what have become your ideas now. The formulae that follow -- the human minds behind them -- call out to you to embody them in your own thoughts, to lend them your mind, and to let them orchestrate the flow of your consciousness, just for the time that your beholding of their presentation takes you. These formulae call you to become them, to “simulate” them in your own inner seeing, to “personify” them, to ‘im-person-ate’ and to ‘im-person-ize’ their intensions and connotations, their meanings, until they have made themselves known to and in you, via the systematic journey of comprehension of modern, standard arithmetic and number along which they are now ready to conduct you.

C.0. 'Dialogue-ic' Model/Translation of Our Dialectical-Ideographic 'Meta-Model' of the Standard Arithmetics' Presentation.

In resonance with the prologue to the quote from Nicholas Rescher extracted in section B. γ. of Part I.: “Already the Socrates of Plato’s *Theaetetus* conceived of inquiring thought as a discussion or dialogue that one carries on with oneself.” [ibid., p. 46], the ‘Dyadic Seldon Function meta-model’ central to this essay can be most directly interpreted -- especially taking into account what we have had to say above regarding “‘personification’” and “‘impersonation’” -- as ‘meta-modeling’ a generic such ‘self-dialogue’, as a readily reusable ‘recipe for thought’ for a systematic and well-ordered refresher for one’s self, and for re-presentation to others, of the contemporary “Standard Arithmetics”. However, in Part I., version 4., we translated our eight-category ‘Dyadic Seldon Function’ model for the systematic dialectic of the Value-Form content in Marx’s *Capital* into the narrative format of an ‘*exo-allo-dialogue*’ between **two** generic, fictional interlocutors, rather than into an ‘*endo-auto-dialogue*’ within the mind of a single such generic, fictional “character”. We can do likewise with our ‘Dyadic Seldon Function meta-model’ presentation for the ‘meta-system’ of the “Standard Arithmetics”, translating it into a somewhat Socratic, ‘dialogue-ic’ “Q&A”, in a way which perhaps more compellingly reveals what is implicitly involved in the Seldon-Function categorial progression algorithm, as applied to this subject-matter. A sample of such a translation is exhibited below. In reviewing the dialogue below, it is important to keep in mind that the “‘analysis of categories’”, referenced repeatedly therein, has a special meaning in this context. The “‘analysis of a category’” means, herein, the process of elaboration of the content of that category, in the sense of the ‘explicitization’ of at least some of its content which would otherwise remain implicit, i.e., if only that category’s name were given: the explicit evocation of some, at least, of the sub-categories of that category, and, perhaps, of some of *their* sub-categories as well, i.e., the elaboration of at least some of the category’s ‘sub-sub-categories’, or ‘sub²-categories’, as well. “‘*Analysis*” in this, dialectical, sense, is the process of human cognition by which we “ascend” [Marx, reversing dialectical predecessors Plato, et al.] to greater detail, or “determinateness” and ‘thought-concreteness’, from greater abstraction and abstract simplicity. It is the oppositely-directed cognitive movement/movement-of-cognition to that of “‘*Synthesis*”’, whereby we “descend” from multiple, more detailed, more specific ‘subⁿ-categories’, to a single, more general, more generic, more “abstract” and simplified ‘subⁿ⁻¹-category’, one that embraces, and [re-]implicitizes into itself, all of the ‘subⁿ-categories’ immediately “‘above’” it, as well as all of those ‘subⁿ⁺¹⁺-categories’ “‘above’” them [if any].

Q1: ¿What is the simplest category that grasps the totality of our, and, in our view, of most people’s ‘untheorized’, ‘unsystematic’, “chaotic” [Marx] experience, and knowledge, of the modern / contemporary Standard Arithmetics?

A1: The category of the “**Natural**” system of arithmetic, the system which is about the **N** set of numbers, and whose analysis is as follows: It is the system that ‘explicitizes’ ‘Standard Numbers’ as “counts” -- as *cardinal* numbers.

Q2: ¿Does the analysis of this arithmetic-system category, $\overset{3}{H}\overset{N}{\#}$, exhaustively systematize our previously fragmentary experience and “chaotic” knowledge of the totality of Standard Arithmetic, accounting entirely for/explaining all of the ‘ideo-phenomena’ of number that we encounter today? ¿Or, are there other categories needed to entirely comprehend that experience and knowledge, to fully classify/systematize and explain those ‘ideo-phenomena’ of number; categories whose content is not explicitly covered by the analysis of the “**Natural Arithmetic**” system-category; categories whose analysis could therefore add to our grasp of the totality of the systems of modern/contemporary “Standard Arithmetic”? We offer into evidence here the **N**-algebraic equation $n + x_1 = n, n \in N$, not satisfiable by any number in **N**: $x_1 \notin N$. And this equation is *paradoxical* from the point of view of the $\overset{3}{H}\overset{N}{\#}$ definition of ‘Standard Number’, of number “**Natural**-ness”, for which addition cannot result in no increase in numeric magnitude.

A2: The category of the arithmetical sub-system of the “‘zeros’”, or of the ‘*ought numbers*’, $a \equiv \{I - I \equiv 0, II - II \equiv 0, III - III \equiv 0, \dots\}$ -- the sub-system category denoted by $\overset{3}{H}\overset{a}{\#}$ -- is needed to satisfy the kind of equation that you cited. Your equation transforms, algebraically, to $x_1 = n - n$, and so invokes the set of results of ‘*self-subtraction*’ for all “**Naturals**”, **n**. The analysis of the category of ‘*the oughts*’ ‘explicitizes’ an expansion of the meaning/definition of ‘Standard Number’, over and above that explicit in the ‘**N**-numbers’ concept: The upshot of this analysis is that not just “counts”, but also ‘no(**n**)-counts’ -- certain kinds of ‘not-counts’ -- are among the ‘Standard Numbers’.

Q3: ¿Do the [sub-]system categories of the “**Naturals**” and the ‘**aughts**’, of “counts” as ‘Standard Numbers’ versus of ‘**no(n)**-counts’ as ‘Standard Numbers’, simply represent two irreconcilable, absolutely disparate and separated and, in some ways, diametrically opposite categories? ¿Or, do we find, in our experience of the human number meme, that these two categories interrelate, and even combine? ¿That is, does the ‘**no(n)**-counts’ category, the counter-example to, and the breakdown of, the “counts” concept/definition of ‘Standard Number’, unite with the category that it counters, to form a new **whole** system-of-arithmetic category? ¿Can our explanation/systematization/classification of the ‘meta-system’ of the “Standard Arithmetics” be adequate as either the analysis of the category $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{N}_{\#}$ by itself, or the analysis of category $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{a}_{\#}$ by itself, as two, distinct, disjunct alternatives?

A3: “Yes”, the “**Naturals**” and the ‘**aughts**’ are two different, separate, disparate, even qualitatively opposite categories of kinds of number, of number ‘ideo-ontology’. But “No”, they are not absolutely separate, or absolutely separable, in our experience of humanity’s numbers meme. And our theory/presentation of the “Standard Arithmetics” ‘meta-system’ is inadequate to that experience if it takes $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{a}_{\#}$ and $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{N}_{\#}$ as only dirempt. Indeed, in our experience/knowledge of modern/contemporary Arithmetics, they both are ingredient in, and inseparable as, the superior -- superior to the Roman Numerals and to the other ancient, additive numeration-systems -- Indo-Arabic numeration system, the place-value numeration system made possible by the advent of **0** as place-holder, and as full number in its own right, in which separate $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{a}_{\#}$ and $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{N}_{\#}$ give way to their “‘complex unity’”, to the $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ system which the former form by their “‘combination’”. The “‘combination’” of the “**Naturals**” and the ‘**aughts**’ constitutes the “**Wholes**” -- the “**Whole** numbers”, such that $\underline{W} \equiv \{0...0, 0...01, 0...02, 0...03, \dots\}$, and in which the equation $\underline{w} + \underline{x}_1 = \underline{w}$, $\underline{w} \in \underline{W}$, which is a “‘well-formed equation’” in $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$, can “now” be satisfied, namely, by $\underline{x}_1 = 0$, $\underline{x}_1 \in \underline{W}$. The analysis of the $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ category yields a concept/definition of ‘Standard Number’ that encompasses and reconciles the “counts” & ‘**no(n)**-counts’ definitions of number, in a single concept/definition of number “‘**Whole**-ness’”, wherein **0** denotes the ‘count-value’ of the ‘**Wholly absent**’.

Q4: ¿Does the analysis of this new, composite, successor system-of-Standard-Arithmetic category, $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ exhaust our experience and knowledge of the totality of contemporary/modern Standard Arithmetic(s)? ¿Does that analysis account entirely for, i.e., does it explain, that experience and knowledge? ¿Or are there other categories that are required to cover, and to systematize, that experience and knowledge; required to classify, and to explain, all of it -- or, at least, most of it: even the most exotic ‘ideo-phenomena’ of modern/contemporary Standard Arithmetic(s) that we have encountered; other categories whose content is not explicitly elaborated by the analysis of the “**Whole**”-numbers arithmetic-system category, other categories whose analysis/‘content-explicitization’ would therefore add to our comprehension of the totality of modern/contemporary Standard Arithmetic(s)? I call your attention, specifically, to the algebraic equation $\underline{w} + \underline{x}_2 = 0$, $0 \neq \underline{w} \in \underline{W}$, which is a “‘well-formed equation’” in $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ [i.e., ‘**w**’, ‘+’, ‘**x**’, ‘**2**’, including as a subscript, ‘=’, & ‘**0**’ are all parts of the “official vocabulary” of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$]. Despite this equation’s being an equation of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ -- a “‘well-formed’”, “legal” equation in $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ [though not so in $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{N}_{\#}$, because $0 \notin \underline{N}$] -- this equation is not *satisfiable* by any number in \underline{W} : $\underline{x}_2 \notin \underline{W}$. This equation defines an ‘ideo-phenomenon’ of ‘decreasive addition’, one which is paradoxical from the point of view of the new, $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$, definition of ‘Standard Number’, because the possibility of ‘decreasive addition’ is not instantiated within \underline{W} . The nature of number as implied and specified in $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \underline{W}_{\#}$ is contradicted by the “number” whose existence is implied by this equation, under the name of its “unknown”, \underline{x}_2 . The behavior ascribed, by this equation as a whole, to the number, \underline{x}_2 , is of a kind-of-number(s) which, when added to any \underline{W} -number, “nullifies” it, decreasing it all the way back to **0**.

A4: The analysis of the $\overset{3}{H}\overset{\#}{W}$ composite category as a whole, and of each of its constituent sub-categories -- $\overset{3}{H}\overset{\#}{N}$, $\overset{3}{H}\overset{\#}{a}$, & $\overset{3}{H}\overset{\#}{Q}$ -- which, taken together, constitute/connote $\overset{3}{H}\overset{\#}{W}$, analyzed again, individually, but now in the context of the

analysis of $\overset{3}{H}\overset{\#}{W}$ as a whole, does further advance our detailed comprehension of our experience and knowledge of the modern/contemporary Standard Arithmetic(s), $\#$. But, all of that analysis still leaves out[side of itself] fundamental aspects of our experience and knowledge of the ‘ideo-phenomena’ of this ‘human-phenomic’ totality that we “‘name’” $\#$. In particular, the category of the axiomatic system-component for the “minus numbers”, $\overset{3}{H}\overset{\#}{m}$, is needed to solve the equation $\mathbf{w} + \mathbf{x}_2 = \mathbf{0}$, in which \mathbf{x}_2 functions as an “additive inverse” of \mathbf{w} , and the analysis of this “minus numbers” category, is needed to advance our comprehension of the modern/contemporary Standard Arithmetic(s) beyond the ken of the $\overset{3}{H}\overset{\#}{W}$ category. The analysis of the axiomatic sub-system category of the “minus numbers”, or of the “negative[ly]-signed” numbers, such that $\mathbf{m} \equiv \{\pm 0 - \mathbf{I} \equiv -1, \pm 0 - \mathbf{II} \equiv -2, \dots\}$, explicitizes the following expansion of the meaning -- of the definition -- of the ‘Standard Numbers’: The upshot of this expansion is that numbers, and that counting, and that “‘counting numbers’”, can have *direction* -- can have one of two “‘co-linear directionalities’”, in terms of the number-“line” convention of the “‘analytic-geometric’” visualization of the “‘number-spaces’” of these number-systems: an explicit *right-hand directionality* [labeled by the ‘+’ sign], or an explicit *left-hand directionality* [labeled by the ‘-’ sign], or, in just a single case -- the case of ± 0 -- directional “‘neutrality’”/‘intra-bi-ality’ [‘ \pm ’].

Q5: ¿Do the [sub-[axioms-]]systems categories of the arithmetic of the “Wholes” & of the arithmetic of the ‘minuses’, of the apparently/superficially ‘undirectional’, “direction-less”, or “mono-directional” counts” as ‘Standard Numbers’ versus of explicitly ‘counter-directional’ counts” as ‘Standard Numbers’, simply form two irreconcilable, absolutely disparate and separated and, in some ways, diametrically opposite categories? ¿Or, do we find, in our experience of the human number meme, that these two categories interrelate, and even combine? ¿That is, does the ‘contra-directional’ counting’ category, $\overset{3}{H}\overset{\#}{m}$, the counter-example to, and ‘demoter’ of, the $\overset{3}{H}\overset{\#}{W}$ concept/definition of ‘Standard Number’, which signals the failure of the $\overset{3}{H}\overset{\#}{W}$ category to constitute the totality of Standard Arithmetic, unite with the category that it counters, demotes, and criticizes, to form a new, *integral* system-of-arithmetic category? ¿Can our sought-after total explanation/systematization/classification of the ‘meta-system’ of the “Standard Arithmetics” be adequate as *either* the analysis of the category $\overset{3}{H}\overset{\#}{W}$ by itself, *or* the analysis of category $\overset{3}{H}\overset{\#}{m}$ by itself, seen as radically disjunct alternatives?

A5: “Yes”, the “Wholes” and the ‘minuses’ are two different, separate, disparate, even qualitatively opposite categories of kinds of number, of number ‘ideo-ontology’. But “No”, they are *not* absolutely separate, nor absolutely separable, in our experience and knowledge of humanity’s numbers meme. And our theory/presentation of the “Standard Arithmetics” ‘meta-system’ is inadequate to that experience if it takes $\overset{3}{H}\overset{\#}{W}$ and $\overset{3}{H}\overset{\#}{m}$ *only* as dirempt. Indeed, we have encountered an axiomatic system of-arithmetic category in which, and by which, the three -- the explicitly-[minus]-signed “minus numbers”, the explicitly-[plus]-signed “plus-numbers”, and, at their meeting point, between them, the explicitly-[double]-signed ‘neutral number’ [± 0] -- are all ‘integ[e]rated’ into a single system of arithmetic, the system of the *integers*, $\overset{3}{H}\overset{\#}{Z}$. Per this system-category, $\overset{3}{H}\overset{\#}{Z}$, we have $\mathbf{Z} \equiv \{\dots, -3, -2, -1, \pm 0, +1, +2, +3, \dots\}$. The explicit three-fold bi-directional classification of numbers is effected in $\overset{3}{H}\overset{\#}{Z}$. That is, each number in \mathbf{Z} has *either* a leftward [–] direction, *or* a rightward [+] direction, *or* no direction at all/neither direction, i.e., both directions at once, with equal “weight” [\pm]. If a pair of ‘oppositely-directioned’ \mathbf{Z} -counts that are also of equal magnitude [of equal “absolute value”] combine additively, they “‘mutually annihilate’” undoing both their directions and their magnitudes, yielding/leaving/arriving at ± 0 : $\mathbf{w} + \mathbf{x}_2 = \pm 0$, if $\mathbf{x}_2 = -\mathbf{w}$. Although it is true that $-\mathbf{w} \notin \mathbf{W}$, it is also true that $-\mathbf{w} \in \mathbf{m} \subset \mathbf{Z}$.

Thus, in this $\overset{3}{H}\overset{\#}{m}$ -and- $\overset{3}{H}\overset{\#}{W}$ -synthesizing axioms-system, the equation $\mathbf{W} + \mathbf{x}_2 = \pm 0$ can be satisfied. The composite axioms-system category $\overset{3}{H}\overset{\#}{Z}$ “‘contains’” seven axioms sub-systems sub-categories: (a.) the 3 past-stages categories $\overset{3}{H}\overset{\#}{N}$, $\overset{3}{H}\overset{\#}{a}$, & $\overset{3}{H}\overset{\#}{a}N$, which, added together, $\equiv \overset{3}{H}\overset{\#}{W}$, which are/is ‘evolutely’ conserved in $\overset{3}{H}\overset{\#}{Z}$, plus (b.) $\overset{3}{H}\overset{\#}{m}$, the core of $\overset{3}{H}\overset{\#}{Z}$, plus; (c.) $\overset{3}{H}\overset{\#}{m}N$, connoting the conversion of the $\overset{3}{H}\overset{\#}{N}$ into a part of the $\overset{3}{H}\overset{\#}{Z}$, plus; (d.) $\overset{3}{H}\overset{\#}{ma}$, connoting conversion of $\overset{3}{H}\overset{\#}{a}$ into another part of $\overset{3}{H}\overset{\#}{Z}$, plus; (e.) $\overset{3}{H}\overset{\#}{ma}N$, connoting the ‘ $\overset{3}{H}\overset{\#}{Z}$ -ization’ of $\overset{3}{H}\overset{\#}{a}N$. We have thusly found our way to a “‘meta-system transition’” [cf. Turchin], $\overset{3}{H}\overset{\#}{W} \rightarrow \overset{3}{H}\overset{\#}{Z}$, a transition within the “‘meta-system’” which is formed by the totality of these Gödelian-Dialectical transitions --

$$\overset{3}{H}\overset{\#}{N} \rightarrow \overset{3}{H}\overset{\#}{W} \rightarrow \overset{3}{H}\overset{\#}{Z} \rightarrow \overset{3}{H}\overset{\#}{Q} \rightarrow \overset{3}{H}\overset{\#}{R} \rightarrow \overset{3}{H}\overset{\#}{C} \rightarrow \overset{3}{H}\overset{\#}{P} \rightarrow \dots$$

-- just as, earlier in this dialogue, we found our way similarly to the “‘meta-system transition’” $\overset{3}{H}\overset{\#}{N} \rightarrow \overset{3}{H}\overset{\#}{W}$.

Q6: ♪Does the analysis of this new, composite, successor system-of-Standard-Arithmetic category, $\overset{3}{H}\overset{\#}{Z}$, exhaustively systematize our experience and knowledge of the totality of contemporary/modern Standard Arithmetic(s)? ♪Does that analysis account entirely for, i.e., does it explain, that experience and knowledge? ♪Or are there other categories, already encountered by us unsystematically, in our “chaotic” experience and fragmentary knowledge of modern/contemporary Standard Arithmetics, that are required to more fully cover, and to more fully systematize, and to render more fully intelligible, that experience and knowledge; required to classify, and to explain, all of it -- or, at least, most of it: even the most exotic ‘ideo-phenomena’ of modern/contemporary Standard Arithmetic(s) that we have ever encountered? ♪Are there other categories whose content is not explicitly elaborated by the analysis of the “*Integers*” arithmetic-system category, $\overset{3}{H}\overset{\#}{Z}$, and of its sub-categories, in the context of the “*Integers*” system-category as a whole? ♪Are there other categories whose analysis/‘content-explicitization’/elaboration would therefore add to/improve/increase the coverage of our theory’s comprehension of [another part of] the totality of the ‘ideo-phenomena’ of modern/contemporary Standard Arithmetic(s)?

.....

C.1. Generic Stage-by-Stage Solution/Presentation Format

The ‘Dialectical Equation’ --

$$\overset{3}{\text{H}}\overset{\#}{\text{H}}_6 = \left(\overset{3}{\text{H}}\overset{\#}{\text{N}} \right)^2_6$$

-- and its ‘syntactically-minimalized’ core --

$$\overset{6}{\text{N}}^2$$

-- concentrate the vast, multi-dimensional meaning of the “‘Standard Number(s)’” and of their “‘Standard Arithmetic(s)’”, up to the $\overset{3}{\text{H}}\overset{\#}{\text{C}} \dashv \oplus \overset{3}{\text{H}}\overset{\#}{\text{h}}$ «*aporia*», by means of ‘semantic-compression/-condensation/-compactification-into-implicitude’.

Our task, in the present, core section of this essay, is, first, to open-up, and then to unpack, that ‘hyper-compactified’ semantic package; to unfold its folded-up ‘semanticities’, to ‘explicitize’ all that is hidden in implicitude within it -- all that it is capable of occulting, by the occultation accomplished by this ‘implicitization’ -- of the meaning of “number”.

We will deploy our stage-by-stage solution-presentation of the above-stated ‘meta-model’ with the aid of the following generic solution-presentation format, applied in, and particularized to, each of its seven stages/steps:

C.1.S. stage/step **S = S**: *descriptive name of stage/step S*.

S.1. *descriptive name of stage*: ...

S.2. stage ‘parametrics’: [total terms count, 2^S] _; [new terms count, 2^{S-1}] _; [new terms needing solution count, $(2^{S-1} - 2)$] _.

S.3. «*aporia*» of this stage: ... $\overset{3}{\text{H}}\overset{\#}{\text{X}} \dashv \oplus \overset{3}{\text{H}}\overset{\#}{\text{g}}_{\text{XX}}$...

S.4. “‘incompleteness’”-revealing “unsolvable” algebraic [“diophantine”] [in]equations-family for this stage: $\dots f(x_s) \gtrless g(x_s) \dots$

S.5. “unsolvable” [in]equations’ paradox: ... *The Paradox of* ...

S.6. shortcut rendition of the ‘product-tion’ of stage S from stage S - 1 [via the ‘meta-meristematic principle’]:

$$\overset{3}{\text{H}}\overset{\#}{\text{H}}_{s+1} = \left(\overset{3}{\text{H}}\overset{\#}{\text{N}} \right)^s \otimes \overset{3}{\text{H}}\overset{\#}{\text{H}}_s$$

S.7. ‘pre-solved’ new terms -- new terms of *given* or *already solved* meanings/definitions:

$$\mathbf{S.7.1.} \equiv \left(\dots \right) \mapsto \boxed{\text{...}}_{2^{s-1}} \dots$$

$$\mathbf{S.7.2.} \equiv \left(\dots \right) \mapsto \boxed{\text{...}}_{2^s} \dots$$

S.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module ζ_1]:

$$\mathbf{S.8.1.} \equiv \left(\dots \right) \mapsto \boxed{\text{...}}_{2^{s-1}+1} \dots$$

The presentation of the stages in this section instantiates a “‘*helical*’”, ‘*qualo-fractal discourse*’, not just via the solution-presentation format, but via the narrative commentaries for each stage as well. Not only each specific stage’s solution-presentation format, but also each specific stage’s narrative commentary, will be kept *as similar as possible* to those of its predecessor stages [if any]. Each such format and narrative commentary will be varied, in its ‘form/content’, only as required by faithfulness to the ‘form/content’ of its specific stage. All other -- unnecessary -- variation will be avoided. This will be attempted, *despite* the escalation of ‘qualitative scale’ with each successor stage; the growth in ‘ideo-ontological’, ‘number-ontological’, ‘arithmetical-ontological’, ‘semantico-syntactic’ content with each stage step -- the higher scale of complexity, of features-richness, of the repleteness of arithmetical ‘ideo-phenomena’, that [a]rises with each unit increment to S, so that this escalation of scale tends to decrease the similitude of each successor step to all of its predecessor steps, the more so the more remote those presentations.

[Note: Spectral coding as used herein, to emphasize *ordinality*, is *relative*, e.g., $\mathbf{N} \equiv \{ \mathbf{I}, \mathbf{II}, \mathbf{III}, \dots \}$, but $\mathbf{W} \equiv \{ \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots \}$].

[Note: In the number-space definitions/specifications of the next sub-section, we switch from a numeral color-coding regime emphasizing ordinality, to a regime indicating the system of arithmetic in which each symbolic element/‘numeral’ so-coded inheres].

[Note: The symbol ‘ \nexists ’, asserting ‘non- \exists istence’, is used, in the next sub-section, to ‘valueate’ an uninstantiated verbal description; an ‘unfulfillable’ specification].

Computational-Shortcut Form of the ‘Dyadic Seldon Function’. The symbol ‘ $\text{---}+$ ’ for us is the total-order *relation-symbol* for purely-qualitative ordinality [the generic quality of ‘*first*-ness’ $\text{---}+$ the generic quality of ‘*second*-ness’ $\text{---}+$...], analogous to the ‘ $<$ ’ total-order relation symbol for the purely-quantitative ordinality of the **N** through the **R**. When applied, *as an operator*, upon a ‘cumulum’ symbol -- ‘ H ’, or ‘ H ’, or ‘ H ’ -- the ‘ $\text{---}+$ ’ symbol commands for (s)election, from the ‘cumulum’ to its right, of its maximal, or ‘meta-meristernal’, term, viz. --

$$\text{---}+ \text{H}_h \equiv \text{---}+ \llbracket \text{Q}_{l_1} \rrbracket^{2^h} \equiv \text{---}+ \llbracket \text{Q}_{l_1} \boxtimes \dots \boxtimes \text{Q}_{l_{2^h}} \rrbracket \equiv \text{Q}_{l_{2^h}} \equiv \underline{\text{max}} \text{H}_h, \text{ so:}$$

$$\text{---}+ \text{H}_h \boxtimes \text{H}_h \equiv \text{Q}_{l_{2^h}} \boxtimes \llbracket \text{Q}_{l_1} \rrbracket^{2^h} \equiv \text{Q}_{l_{2^h}} \boxtimes \llbracket \text{Q}_{l_1} \boxtimes \dots \boxtimes \text{Q}_{l_{2^h}} \rrbracket \equiv \llbracket \text{Q}_{l_1} \boxtimes \dots \boxtimes \text{Q}_{l_{2^{h+1}}} \rrbracket \equiv \text{H}_{h+1}.$$

This shortcut form works, in part, because, per the ‘*«aufheben»* evolve product rule’, Axiom §9, of the **Q** axioms-system of dialectical arithmetic, the operand, or “multiplicand”, is returned again as part of the product, added-back into that product, added to the ‘new ontology term(s)’ in that product. So the ‘ontological multiplication’ $\text{---}+ \text{H}_h \boxtimes \text{H}_h$ delivers *all of* H_h as *part of* the product/result of that multiplication. That is, the operation of the final term of H_h -- i.e., of $\text{Q}_{l_{2^h}}$, the ‘vanguard term’, or ‘meta-meristernal term’, of H_h -- upon that very ‘cumulum’, H_h , of which it, $\text{Q}_{l_{2^h}}$, is also a part, as H_h ’s final term, reproduces that entire ‘cumulum’, H_h , as well as yielding the “‘*real subsumption*’”, by that final term, $\text{Q}_{l_{2^h}}$, of all prior terms in that H_h ‘cumulum’, a series of subsumptions which is capped by the ‘*self*’-subsumption of that ‘vanguard term’, $\text{Q}_{l_{2^h}}$, itself, and, thereby, capped by the irruption of the new, successor ‘vanguard term’/‘meta-meristernal term’, $\text{Q}_{l_{2^{h+1}}} \equiv \text{Q}_{l_{2 \cdot 2^h}} \equiv \text{Q}_{l_{2^1 \cdot 2^h}} \equiv \text{Q}_{l_{2^h \cdot 2^1}}$, the successor ‘contra-thesis term’ to $\text{Q}_{l_{2^h}}$ [**N** arithmetic survives at subscript level in **Q**].

The “full regalia” form of the Dyadic Seldon Function’s generic ‘purely-ideo-ontological’ ‘[Meta-]Systematic Dialectic Equation’, or ‘Dialectical [Meta-]Evolution Equation’ -- the **Q**-algebraically nonlinear, transcendental Dialectical Equation to which the ‘Dyadic Seldon Function’ is the general solution -- is:

$$\text{H}_{s+1}^x = \text{H}_s^x \otimes \text{H}_s^x.$$

That “full regalia”, more general rendition is important algorithmically, & even substantively, e.g., for some alternative Axiom §9 product rules, such as for some of the “‘Gödelian’” variants of Axiom §9, which use “Gödel number”-like subscript rules to make the “‘ancestry’” of each term ‘decodable’, and also to render more explicitly the ‘intra-duality’ of each “‘non-Gödelian’” ‘*«aufheben»* evolve product rule’ term, by explicitly exhibiting separate terms for ‘retrograde/-degenerative conversions’; ‘explicitizing’ qualitatively-unequal values for the commutations of the each “‘hybrid’” term’s subscripts. But from a ‘calculational-efficiency’ standpoint, when using “‘non-Gödelian’” Axiom §9, most of the product terms yielded by the equation immediately above are redundant. We therefore use the following, shortcut, “‘fast’” algorithm in all sections **S.6**, the one that “cuts to the chase” for getting from one step/stage to the next [given *«arché»* $\equiv \text{H}_x^x$]:

$\text{H}_{s+1}^x = \text{---}+ \text{H}_s^x \otimes \text{H}_s^x = \text{H}_s^x \otimes \text{H}_s^x$. In words, the ‘meta-meristernal rule’ is: ‘*To obtain the next cumulum, multiply the given cumulum by its own final, ‘meta-meristernal’ term.*’ Given the afore-defined format(s), we are now ready to deliver a stage-by-stage, term-by-term presentation/solution of our ‘meta-model’.

C.2. Stage-by-Stage/Term-by-Term Solution/Presentation of the Standard Arithmetics' Dialectical 'Meta-Model'

C.2.0. stage/step **S = 0**: Starting Point -- The System of Arithmetic of the "Standard **N**atural Numbers", the **N**.

- 0.1. descriptive name of stage: «**Arché**» [Beginning, Governing Source, Starting-Point, Point-of-Departure, Ever-Present Origin].
- 0.2. stage 'parametrics': [total terms #, 2^0] 1; [new terms #, 2^{0-1}] $\notin \mathbf{W}$; $\therefore \nexists$; [new terms needing solution #, $(2^{0-1} - 2)$] $\notin \mathbf{W}$; $\therefore \nexists$.
- 0.3. «**aporia**» of this stage: \nexists .
- 0.4. "incompleteness"-revealing "unsolvable" algebraic ["diophantine"] equations-family specific to this stage: \nexists .
- 0.5. "unsolvable" equations' paradox: \nexists .
- 0.6. shortcut rendition of the 'product-tion' of stage 0 from stage 0-1 [via the 'meta-meristematic principle'] [$-1 \notin \mathbf{W}$]: $\therefore \nexists$.
- 0.7. 'pre-solved' new terms -- new terms of *given* or *already solved* meanings/definitions:
 - 0.7.1. $\equiv \langle \langle \overset{3}{\mathbf{N}}_{\#} \rangle \rangle \vdash \mathbb{Q}_{\perp_1}$ -- the first-and-second-order-logic expression of the Dedekind/Peano axioms-system of the arithmetic of the "**N**atural" Numbers, **N** $\equiv \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \dots\}$.
- 0.8. stage-specific solution, term-by-term [using 'Organonic Algebraic Method', Procedure Module ζ .]: \nexists .

Number-Line Cumulative Progression Analytic-"Geometric" Model
of the **Gödelian Dialectic of the Standard Arithmetics**: stage s = 0

s = 0

$\langle \langle \mathbf{N} \rangle \rangle^{2^s} = \langle \langle \mathbf{N} \rangle \rangle^{2^0} = \langle \langle \mathbf{N}^1 \rangle \rangle = \langle \langle \mathbf{N} \rangle \rangle$

N \Leftrightarrow **I** **II** **III** ...

Narrative Commentary for stage $S = 0$. The known portion of the story of the development to, and of the birth of, the “**N**atural” Numbers, as a practical human-phenomic reality, and, later, as a formalized theory [which we denote by $\overset{3}{H}\underline{N}_{\#}$], in itself already forms a fascinating and foundational facet in the psychohistorical dialectic of the self-development of the Terran ‘Human Phenome’. Suffice it to say that the so-called “Natural” Numbers are anything but “Natural” in the sense of having been “‘given to humanity ready-made by pre-human nature’”. However, because, in the present study, we are taking $\overset{3}{H}\underline{N}_{\#}$ as our «*arché*», our *first* beginning, the consideration of the genesis of $\overset{3}{H}\underline{N}_{\#}$ itself is beyond the scope of this essay. For the reader interested in the [psycho]historical genesis of the **N**, we recommend especially volume **I** of the multi-volume series ***BEFORE WRITING***, the volume entitled ***From Counting to Cuneiform***, by Denise Schmandt-Besserat [U. of Texas Press, Austin, **1992**]. In that initial volume, Dr. Schmandt-Besserat provides a “‘psychohistorical’” theory of the protracted development, in ancient Babylonia, of written numerals, initially as a [what we term] “‘tokenography’”, from out of a prior system of [what we term] “‘tokenology’”, one that used ‘effigetic’ fired clay tokens to record, initially tithe-like or tribute-like obligatory gifts of goods to the temples of the redistributionist Priesthood’, & later, also commodity-exchange transactions, for a much-expanded ontology of “goods”, & that ties the origin of number-writing to the origin of writing in general. A [psycho]historical-/meta-systematic dialectical mathematical ‘meta-model’ of this [psycho]history is to be presented in volume **2** of ***A Dialectical Theory of Everything***, Chapter **+7.03: *Dialectical ‘Meta-Model’*** -- ***The [Psycho]Historical Dialectic of the Genesis of Written Language***.

Herein, we define the “**N**atural” Numbers as $\mathbf{N} \equiv \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \dots\}$, rather than as $\{1, 2, 3, \dots\}$, or as $\{0, 1, 2, 3, \dots\}$. This is a nod to Terran *psychohistory*, despite our commitment that our theory, encapsulated in $\overset{3}{H}\underline{H}_{\#}^6 = (\overset{3}{H}\underline{N}_{\#})^6$, shall be limited to an essentially ‘synchronic theory’, of the *contemporary* “‘meta-system’” of the “Standard Arithmetics”. For the past is ingredient in the present, «*aufheben*»-conserved/elevated/negated, together with the seeds of the future. After all, the Roman Numerals are still, ‘evolutely,’ in some vestigial use at the present time. And the conceptual «*travail*» of the Occidental branch of Terran humanity -- if *not* of, e.g., the Asian, and the Mayan, branches -- in arriving at **0** as a full number, an ordeal still present in the «*aporia*» of division by **0**, was too great not to leave some etchings and traces in our *psychohistorical*-dialectical ‘meta-model’ of the Gödelian Dialectic. Herein, stage/step **S = 0**, and its sole category, $\overset{3}{H}\underline{N}_{\#}$, are taken as the simplest, as the most immediate, as the most abstract relative to the fullest measure of the richness of ‘number phenomena’ extant in contemporary “Standard Arithmetics”, of all of the stages/steps of our journey, and of our story; as the deepest root of, and therefore as the point-of-departure for, our ordinal/systematic/taxonomic exploration of ‘the Standard Numbers’, using the algebraic language of the \mathbf{N}^Q ‘*non*’-Standard Numbers’.

The “‘extension’” of the “‘idea-object’” “intended”, or “connoted”, by the “intensional symbol” $\overset{3}{H}\underline{N}_{\#}$, is rendered next below.

[Note on Axioms-Systems Representations, below: Not all of the sentences of the axioms-systems of arithmetic represented below are “independent” axioms. Some are key theorems, in the sense that some of these sentences can be derived deductively from subsets of the others. We have included such “redundant” axiom-sentences because our goal here is not to display the minimal system of such sentences from which all other desired sentences can be deductively derived as theorems, but is rather to more fully characterize each axioms-system intuitively for the reader. These lists of arithmetical sentences are therefore better characterized as presenting systems of fundamental arithmetical “‘rules’” -- each presents the core “‘rules-system’” of the given system of arithmetic.]

What the Symbol $\overset{3}{H}\underset{\#}{N}$ "Intends": *Axioms-System of the Arithmetic of the so-called "Natural" Numbers, N* [commenced]

NØ. N is not the Empty Set.

$N \neq \emptyset \equiv \{ \};$

First Order Peano Postulates: [mainly] "Phonogramic" Rendering
NP1. **1** is a "Natural" Number.

[mainly] "Ideogramic" Rendering
1 $\in N$;

NP2. The successor of a "Natural" Number is also a "Natural" Number.

$n \in N \Rightarrow s(n) \in N$;

NP3. No two "Natural" Numbers have the same successor.

$n, m \in N \ \& \ n \neq m \Rightarrow s(n) \neq s(m)$;

NP4. **1** is not the successor of any "Natural" Number.

$\neg \exists x \in N \mid s(x) = 1$;

Second Order Generalized Peano *Mathematical Induction* Postulate, for "Natural" Arithmetic, for any Predicate, **P**

NP5. $[\forall P][\forall n_1, n \in N][[[[P(n_1)] \ \& \ [P(n_1)] \Rightarrow [P(s(n_1))]]] \Rightarrow [[\forall n \in N][n \geq n_1][P(n)]]]$;

First Order Axioms of "Natural" **Addition**:

NA1. For all n_1, n_2 in N , $n_1 + n_2 = n_2 + n_1$ [**Additive Commutativity**]; $[\forall n_1, n_2 \in N][n_1 + n_2 = n_2 + n_1]$;

NA2. For all n_1, n_2, n_3 in N , $(n_1 + n_2) + n_3 = n_1 + (n_2 + n_3)$ [**Additive Associativity**];

$[\forall n_1, n_2, n_3 \in N][(n_1 + n_2) + n_3 = n_1 + (n_2 + n_3)]$;

NA3. For all n_1, n_2 in N , $n_1 + n_2$ is in N [**Additive Closure of N**]; $[\forall n_1, n_2 \in N][n_1 + n_2 \in N]$;

First Order Axioms of "Natural" **Multiplication**:

NM1. For all n_1, n_2 in N , $n_1 \times n_2 = n_2 \times n_1$ [**Mult. Commutativity**]; $[\forall n_1, n_2 \in N][n_1 \times n_2 = n_2 \times n_1]$;

NM2. For all n_1, n_2, n_3 in N , $(n_1 \times n_2) \times n_3 = n_1 \times (n_2 \times n_3)$ [**Multiplicative Associativity**];

$[\forall n_1, n_2, n_3 \in N][(n_1 \times n_2) \times n_3 = n_1 \times (n_2 \times n_3)]$;

NM3. For all n_1, n_2 in N , $n_1 \times n_2$ is in N [**Multiplicative Closure of N**]; $[\forall n_1, n_2 \in N][n_1 \times n_2 \in N]$;

NM4. There is an element **1** in N such that, for all n in N , $1 \times n = n$ [**Multiplicative Invariance Element**];

$[\exists 1 \in N][\forall n \in N][1 \times n = n \times 1 = n]$;

What the Symbol $\overset{3}{H}\underset{\#}{N}$ "Intends" [continued & concluded]

First Order Axioms for the "Hybridization" of the "Natural" Operations of "Multiplication" and of "Addition":

NH1. For all n_1, n_2, n_3 in N , $(n_1 + n_2) \times n_3 = n_1 \times n_3 + n_2 \times n_3$;

$[\forall n_1, n_2, n_3 \in N][(n_1 + n_2) \times n_3 = n_1 \times n_3 + n_2 \times n_3]$ [**Distributivity** ["Hybridization"] of **Addition** over **Multiplication**];

NH2. For all n_1, n_2, n_3 in N , $n_1 \times (n_2 + n_3) = n_1 \times n_2 + n_1 \times n_3$;

$[\forall n_1, n_2, n_3 \in N][n_1 \times (n_2 + n_3) = n_1 \times n_2 + n_1 \times n_3]$ [**Distributivity** ["Hybridization"] of **Multiplication** over **Addition**];

First Order Axioms of "Natural" **Exponentiation**:

NE1. For all n_1, n_2, n_3 in N , $n_1^{n_2} \times n_1^{n_3} = n_1^{n_2 + n_3}$ [**Exponentiation Additivity**];

NE2. For all n_1, n_2, n_3 in N , $(n_1^{n_2})^{n_3} = n_1^{n_2 \times n_3}$ [**Exponentiation Multiplicativity**];

NE3. For all n_1, n_2, n_3 in N , $(n_1 \times n_2)^{n_3} = n_1^{n_3} \times n_2^{n_3}$ [**Exponentiation "Distributivity"**];

NE4. For all n_1, n_2 in N , $n_1^{n_2}$ is in N [**Exponentiation Closure**];

NE5. For all n in N , $n^1 = n$ [**Exponentiation Invariance**];

Axioms of "Natural" **Order**:

NO1. For all n_1, n_2 in N , either $n_1 > n_2$, or $n_1 = n_2$, or $n_1 < n_2$ [**Trichotomy "Law" [First Order]**];

$[\forall n_1, n_2 \in N][[n_1 > n_2] \vee [n_1 = n_2] \vee [n_1 < n_2]]$;

NO2. For all n_1, n_2 in N , $n_1 + n_2 > n_1, n_2$ [**Additive Order [First Order]**];

$[\forall n_1, n_2 \in N][n_1 + n_2 > n_1, n_2]$;

NO3. For all n_1, n_2 in N , $n_1 \times n_2 > n_1, n_2$ [**Multiplicative Order [First Order]**];

$[\forall n_1, n_2 \in N][n_1 \times n_2 > n_1, n_2]$;

NO4. Any non-empty set of "Natural" Numbers contains a *least* element [**Well Ordering Principle [Second Order]**];

$[\forall S \subseteq N][[S \neq \emptyset][\exists s \in S][\forall n \in S][n \leq s]]$.

C.2.1. stage/step $s = 1$: The system-fragment of ‘**aught** numbers’, **a**, formally subsumes the “‘Standard **N**aturals’” system.

1.1. descriptive name of stage: *The “‘**Formal Subsumption**’” of the “**Natural Numbers**” by the ‘**aught Numbers**’.*

1.2. stage ‘parametrics’: [total terms count, 2^1] **2**; [new terms #, 2^{1-1}] **1**; [new terms needing solution #, $(2^{1-1} - 2)$] $\notin \mathbf{W}$; $\therefore \nexists$.

1.3. «**aporia**» of this stage: $\frac{3}{H-\#} \mathbf{N} \vdash \oplus \frac{3}{H-\#} \mathbf{a}$.

1.4. “‘incompleteness’”-revealing “unsolvable” algebraic [“diophantine”] equations-family for this stage: $n + x_1 = n \mid n \in \mathbf{N}$.

1.5. “unsolvable” equations’ paradox: *The Paradox of ‘Non-Increasing’ Addition within $\frac{3}{H-\#} \mathbf{N}$.*

1.6. shortcut rendition of the ‘product-tion’ of stage **1** from stage **0** [via the ‘meta-meristematic principle’]:

$$\begin{aligned} \frac{3}{H-\#} \mathbf{H} \bigg|_1^{\#} &= \frac{3}{H-\#} \mathbf{N} \otimes \frac{3}{H-\#} \mathbf{H} \bigg|_0^{\#} = \frac{3}{H-\#} \mathbf{N} \otimes \left(\frac{3}{H-\#} \mathbf{N} \right) = \frac{3}{H-\#} \mathbf{N} \left(\frac{3}{H-\#} \mathbf{N} \right) = \frac{3}{H-\#} \mathbf{N}^2 = \\ &= \frac{3}{H-\#} \mathbf{N} \oplus \Delta \left(\frac{3}{H-\#} \mathbf{N} \right) = \frac{3}{H-\#} \mathbf{N} \oplus \frac{3}{H-\#} \mathbf{a} = \frac{3}{H-\#} \mathbf{N} \vdash \oplus \frac{3}{H-\#} \mathbf{a}. \end{aligned}$$

1.7. ‘pre-solved’ new terms -- new terms of *given* or *already solved* meanings/definitions:

1.7.1. $\equiv \left(\frac{3}{H-\#} \mathbf{a} \right) \vdash \Rightarrow \mathbb{I}_2$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

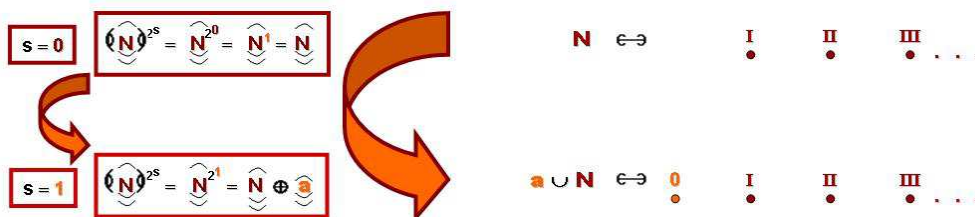
‘sub-arithmetic’ of the ‘**aught Numbers**’, $\mathbf{a} \equiv \{ \mathbf{I} - \mathbf{I} \equiv \mathbf{0}, \mathbf{II} - \mathbf{II} \equiv \mathbf{0}, \mathbf{III} - \mathbf{III} \equiv \mathbf{0}, \dots \}$, the sub-space of the **W** number-space containing all ‘self-differences’ [all ‘self-subtractions’] of “**Natural**” Numbers, $\mathbf{N} \equiv \{ \mathbf{I}, \mathbf{II}, \mathbf{III}, \dots \}$; fruition of the [immanent]

self-critique of $\frac{3}{H-\#} \mathbf{N}$, namely $\frac{3}{H-\#} \mathbf{N} \otimes \mathbf{N} \equiv \frac{3}{H-\#} \mathbf{a}$.

1.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module ζ_1]: \nexists .

Number-Line Cumulative Progression Analytic-“Geometric” Model
of the **Gödelian Dialectic of the Standard Arithmetics**:

Transition from $s = 0$ to $s = 1$ -- from «**arché**» to first «**aporia**»



Narrative Commentary for stage S = 1. The Opposite of the **N**. The “Natural” Numbers are, on the face of them, explicitly, the cardinal numbers, the counting numbers, and their arithmetic is the arithmetic of simple counting. Each “Natural” Number is an abstract, generic count -- a certain aggregate of unit counts, each one denoted by **1** -- and is so abstract that there is no vestige, no shadow left of the thing(s) that are counted by it, no explicit mention even of a generic such thing counted, left in them, in their meaning [semantics], or in their notation [syntax]. However, connotations about those qualitative things counted remain implicit in them, however suppressed. Such a “qualitative thing”, in itself, is of the kind of ‘non-counts’; one of entities upon which counting must yet be performed, to render them “quantified”. Thus the «genos» category of the «species» sub-categories of the ‘non-counts’ is the immanent “‘Janus-face’”, the implicit, inner dual, the occulted/hidden internal opposition to the outer, explicit face of the **N**, whose arithmetic is codified in the axioms-system category $\frac{3}{H}N_{\#}$. If we ‘explicitize’ this

implicit, hidden ‘inner dual’ of the **N** as an explicit, ‘outer dual’, via our minds -- as $\frac{3}{H}N_{\#}(\frac{3}{H}N_{\#})$, “‘personifying’” & “‘impersonating’” & ‘subjectifying’ $\frac{3}{H}N_{\#}$ -- and reflecting upon our minds’ own objectified past content as such, as $\frac{3}{H}N_{\#}(\frac{3}{H}N_{\#})$, then we arrive at the ‘self-reflexion’, or ‘self-reflexive function/operation’, connoted by $\frac{3}{H}N_{\#}(\frac{3}{H}N_{\#})$, as generating the ‘antithesis-sum’ of/between the arithmetic of the **N** and that of their ‘outerized’ inner dual, the now ‘outer dual’, which we denote, generically, by $\Delta(\frac{3}{H}N_{\#})$ --

$$\text{as } s = 0 \rightarrow s = 1, \text{ so also does } \frac{3}{H}N_{\#} \rightarrow \frac{3}{H}N_{\#}(\frac{3}{H}N_{\#}) = \frac{3}{H}N_{\#} \oplus \Delta(\frac{3}{H}N_{\#}).$$

Therein, $\Delta(\frac{3}{H}N_{\#})$ must have to do with at least one «species» sub-category of the «genos» category of the ‘non-counts’. We can also arrive at the «aporia» posited above by reasoning from the dialectical norm that the second category in any “systematic”-[or ‘meta-system-atic’]-dialectical categorial progression is typically an opponent of, or a qualitative opposite to, its first category, & by asking ourselves “‘What is the opposite of a count?’”, & then by answering our ‘self-question’ via the ‘self-answer’ “‘a non-count.”, or “‘a no-count.”. But, if the **N** -- the “counts” -- are numbers, must any ‘no[n]-count’ also therefore be a non-number? Or is there at least one kind of ‘no[n]-counts’ which, too, can be a kind of number(s)?

Is there a kind of arithmetic, based upon a kind of number(s), which is opposite, in kind, to the kind of the “Natural” Numbers -- not just opposite to any one specific “Natural” Number or another, but to the “Natural” Numbers as a whole; as a totality -- opposite to the whole “‘genre’” of what it is to be, generically, a “Natural” Number?

The Equation, “‘Well-Formed’” within $\frac{3}{H}N_{\#}$, that is Not Solvable within $\frac{3}{H}N_{\#}$, and that Therefore Points, from within $\frac{3}{H}N_{\#}$, to beyond $\frac{3}{H}N_{\#}$ -- to within $\frac{3}{H}W_{\#}$.

There is an equation which epitomizes this whole dialectic of $\frac{3}{H}N_{\#}$ -- the outbreak of ‘self-antithesis’ within the **N**. This equation is: $n + x_{\alpha} = n$ [for all $n \in \mathbf{N}$].

This equation presents a “paradox of addition” for the “Natural” Numbers «mentalité»/ideology about number-“addition”, and about “number” in general, that can be stated as follows: [“Natural”] addition always produces an increase in number. Therefore: How can there be a kind of number -- the “type” of the “unknown”, x_{α} , of this equation -- by which addition produces no increase? This equation is “‘well-formed’” within $\frac{3}{H}N_{\#}$, in that all of its symbolic elements, ‘n’, ‘+’, ‘x’, ordinal subscript ‘ α ’, and ‘=’, as well as the syntax of their ‘inter-mutually-related’ arrangement in this equation, are official, “legal”, ‘define-able’, and “meaningful”, within the semantic rules, and within the syntactic rules, of the “‘rules-system’” of $\frac{3}{H}N_{\#}$. Therefore, we now may ask ourselves: Furthermore, and more specifically, is

there a kind of number(s), somewhere in our experience of contemporary ‘Standard Number’, that is embedded within a non- $\frac{3}{H}N_{\#}$ [sub-]system of arithmetic, such that this kind of number(s) can solve this algebraic equation, and thereby also resolve the paradox -- from the point of view of ‘number “Natural”-ness’ -- that this equation expresses, this equation which is immanent to/“‘well-formed within’”, the “Natural” Numbers system of arithmetic, but whose algebraic “unknown”, x_{α} , nonetheless cannot be satisfied or expressed by any “Natural” Number in the entire, potentially-infinite “Natural” Numbers language of $\frac{3}{H}N_{\#}$?

A standard algebraic manipulation of this equation into an equivalent algebraic equation -- $[n + x_{\alpha} = n] \Rightarrow [n + x_{\alpha} - n = n - n] \Rightarrow [x_{\alpha} = n - n]$ -- will explicitly reveal that, if there is a kind of number(s) that can solve this equation, it must be a kind that can be constructed by ‘self-subtraction’, but, within **N**, if we rule that the presence of an **n** subtracted from an **n** on any single side of an $\frac{3}{H}N_{\#}$ -algebraic equation is replaceable by __ [blank], then this only leads us to the ‘aporic’ arithmetical/non-algebraic equation $n - n = _$ [non-algebraical given that **n** generically denotes a single known arithmetical constant value from **N** in any given case, so that there is no algebraical “unknown” in this therefore purely-arithmetical equation], i.e., to the ‘non-expression’ of an apparent $\frac{3}{H}N_{\#}$ -‘non-expressible’.

This expressive failure only further establishes that the deficiency of $\frac{3}{H}N_{\#}$ with regard to this equation -- that the ‘inexpressibility’, in the $\frac{3}{H}N_{\#}$ -language, of the solution to this equation -- is an immanent deficiency of the $\frac{3}{H}N_{\#}$ system -- a deficiency on its own terms, not on any alien terms, and a deficiency within $\frac{3}{H}N_{\#}$ that calls for a self-critique -- an immanent critique -- of [our minds “‘impersonating’”] $\frac{3}{H}N_{\#}$, by [our minds “‘impersonating’”] $\frac{3}{H}N_{\#}$.

In our collective experience of contemporary “Standard Arithmetics”, in every stage beyond their «*arché*» stage, $\frac{3}{H}N_{\#}$, purely-quantitative ‘no[n]’-counts’ most decidedly are numbers in their own right. They constitute a most important and useful kind of number(s), the kind that is key to the superlative supersession of ancient arithmetical notations by the medieval Indo-Arabic “place-value” numerals system, involving 0 as “place-holder”. The Indo-Arabic numeral ‘0’ denotes the universality of ‘the no[n]’-count numbers’; the group-name, or «*species*»-name, for all of its logical-individual instances, I – I, II – II, III – III, etc. But, together with all of its advantages, the incremental ‘ideo-ontology’ of 0 brings with it, to this day, a new, and still unrelieved «*aporia*», that of division by 0, an «*aporia*» to which none of the “Standard Arithmetics”, modeled, in part, herein, provide the Gödelian-Dialectical solution.

Let us symbolize/name/connote the axioms-[sub-]system of arithmetic for the domain of ‘non-count numbers’, conceived, initially, as a radically separate domain from that of the “count numbers” of N, in ‘mnemonization’ of the word “aught”, which means “zero, cipher, or aught”, i.e., by the term: $\frac{3}{H}a_{\#}$, such that --

$a_{\#} \equiv$ the solution-set for x_{α} , restricted to only those “aughts” ‘constructable’ by ‘self-subtraction’ of “Natural” Numbers $\equiv \{I - I, II - II, III - III, \dots\}$.

Let us therefore re-begin our dialectic of the “Standard Arithmetics”, by a second beginning, in stage/step S = 1, with, and from, the explicitly recognized opposition between -- with the “antithesis” of -- the arithmetic of counts, versus an ‘arithmetic of non-counts’. That is, we are now ready to more-specifically re-formulate the formula given above, for the first part -- the ‘antithesis-sum’ part -- of the dialectic of $\frac{3}{H}N_{\#}$, as follows --

$$\text{as } s = 0 \rightarrow s = 1, \text{ so also does } \frac{3}{H}N_{\#} \rightarrow \frac{3}{H}N_{\#} \left(\frac{3}{H}N_{\#} \right) = \frac{3}{H}N_{\#} \neg \oplus \frac{3}{H} \bigcirc_{NN}^{\#} \equiv \frac{3}{H}N_{\#} \neg \oplus \frac{3}{H}a_{\#}.$$

The second, final part of the final sum above, ‘ $\dots \neg \oplus \frac{3}{H}a_{\#}$ ’, is the ‘fruition of the immanent-/self-/critique of $\frac{3}{H}N_{\#}$ ’, which is also more generically connoted by ‘ $\dots \neg \oplus \frac{3}{H} \bigcirc_{NN}^{\#}$ ’. The first, initial part of the final sum above, ‘ $\frac{3}{H}N_{\#} \neg \oplus \dots$ ’, is ‘the [“evolute”] conservation of the object of critique’.

The Dialectic of “Natural” Number as Example of the Dialectical Cognition of Transition within the Totality of the Contemporary “Standard Arithmetics” Generally.

In general, $\frac{3}{H}X_{\#} \left(\frac{3}{H}X_{\#} \right)$ connotes the ‘contra-Boolean’ “‘[s]election’” of $\frac{3}{H}X_{\#}$ ’s own, suppressed ‘intra-dual’, or ‘self-dual’, from out of the implicit ‘in-side’ of $\frac{3}{H}X_{\#}$ itself, and the ‘de-suppression’ of that dual, the ‘outering’ of it; the ‘externalization’ of it, the ‘explicitization’ of it, to standing on an equal footing with $\frac{3}{H}X_{\#}$ itself; the “‘addition’”, to $\frac{3}{H}X_{\#}$ itself, of that formerly implicit, internal, hidden, suppressed dual, within $\frac{3}{H}X_{\#}$ itself, as a now apparent, explicit, external other to/- external dual of $\frac{3}{H}X_{\#}$ --

$$\text{as } s \rightarrow s + 1, \text{ so also does } \frac{3}{H}X_{\#} \rightarrow \frac{3}{H}X_{\#} \left(\frac{3}{H}X_{\#} \right) = \frac{3}{H}X_{\#} \neg \oplus \bigoplus \left(\frac{3}{H}X_{\#} \right)$$

At the heart of the above is the ‘contra-Boolean’ equation $\frac{3}{H}X_{\#}^2 = \frac{3}{H}X_{\#} \oplus \bigoplus \left(\frac{3}{H}X_{\#} \right)$, in which $\bigoplus \left(\frac{3}{H}X_{\#} \right)$ connotes a kind of return of the repressed of $\frac{3}{H}X_{\#}$. Were we to count each distinct family of N-unsolvable diophantine equations, “‘well-formed’” within $\frac{3}{H}N_{\#}$, as separate ‘intra-opponents’ of $\frac{3}{H}N_{\#}$, then we would characterize $\frac{3}{H}N_{\#}$, not as an ‘intra-duality’, or ‘self-duality’, but, rather, as an ‘intra-multality’ or ‘self-multality’.

Once one stops pretending that higher-order logic axiomatizations of the “Natural” Numbers arithmetic are perfect -- e.g., are self-consistent and complete, as if Gödel’s First Incompleteness Theorem was still unknown, then the ‘contra-Boolean’ expression --

$$\frac{3}{H}N_{\#}^2 = \frac{3}{H}N_{\#} \oplus \bigoplus \left(\frac{3}{H}N_{\#} \right)$$

-- suddenly makes tremendous sense. It expresses [the first part of] ‘The Gödelian Dialectic’ of the “Natural” Numbers system of arithmetic.

The mind of a human subject -- i.e., the minds of you and of I, of each of us who is following this Seldon Function formula-programmed 'self-dialogue' for her-/himself, as a person living and experiencing in the contemporary context, in the aftermath of the advent of Gödel's theorems -- '[re-]subjectifies', in effect, the objectified, documented higher-order logic axioms-system that, as objective/observed/beheld, we connote by ' $\underline{\underline{3N}}_{H\#}$ '. That is, such a human subject 'mentally embodies'

$\underline{\underline{3N}}_{H\#}$, "'personifies'" and "'impersonates'" that arithmetic, identifies her/his mind with the meaning/living spirit of that system, "'knows'" it, in an act of cognition which we connote by ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{\quad}})$ '. That human subject -- in that spirit, simulating ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{\quad}})$ ' [as 'mentally incarnating'/'mentally embodying' ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{\quad}})$ '] -- then reflects upon, "'beholds'", assesses & evaluates, $\underline{\underline{3N}}_{H\#}$ -- as objective/objectified/dead system of arithmetic, ' $\underline{\underline{3N}}_{H\#}$ ', e.g., as paper documentation thereof, or even as still-remembered past content-of-mind/state-of-mind/state-of-thought -- from the standards and viewpoint of $\underline{\underline{3N}}_{H\#}$ itself: 'contrasts $\underline{\underline{3N}}_{H\#}$ with itself' -- itself as subject / "'spirit of its law'" versus itself as object / "'letter of its law'". That act of cognition precipitates a "'subject/object identical moment'", connoted by the combination of the $\underline{\underline{2}}$, $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$, from which the fruition of this 1st self-critique of $\underline{\underline{3N}}_{H\#}$ emerges: the $\underline{\underline{3a}}_{H\#}$ term of $\underline{\underline{3N}}_{H\#} \oplus \underline{\underline{3N}}_{H\#}$ $\equiv \underline{\underline{3N}}_{H\#} \oplus \underline{\underline{3a}}_{H\#}$. That is, the self-reflection of $\underline{\underline{3N}}_{H\#}$, as $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#}) \equiv \underline{\underline{3N}}_{H\#}^2$, turns up a 'self-discrepancy': the exclusion, from the explicit $\underline{\underline{3N}}_{H\#}$ concept of "counting number arithmetic", of 'no[n]-counts' as numbers; of the arithmetic of 'no[n]-counts', hence divulging the "'incompleteness'" of the $\underline{\underline{N}}$ concept of number, and thus of the system $\underline{\underline{3N}}_{H\#}$ as a whole.

We epitomize this divulgence by highlighting the "'well-formed immanence'" of the family of algebraic equations $\{[n + x_\alpha = n \mid n \in \underline{\underline{N}}]\}$, or, equivalently, of $\{x_\alpha = n - n \mid n \in \underline{\underline{N}}\}$, which express a paradox for the $\underline{\underline{3N}}_{H\#}$ -native notion of the nature and meaning/definition of number-in-general, including of number addition-in-general. This is a family of equations which is 'formable' in $\underline{\underline{3N}}_{H\#}$, but which is nonetheless unsolvable in $\underline{\underline{3N}}_{H\#}$, because $\underline{\underline{3N}}_{H\#}$ lacks/excludes the a 'no[n]-count' number(s), denoted generically by 0.

That mind, achieving this immanent critique of $\underline{\underline{3N}}_{H\#}$, then rectifies this lack, revealed as a result of that critique, by appending $[\oplus]$ a corrective, a supplement, an amendment -- $\Delta(\underline{\underline{3N}}_{H\#}) \equiv \underline{\underline{3a}}_{H\#} \mid a \equiv \{I - I, II - II, III - III, \dots\}$ -- to $\underline{\underline{3N}}_{H\#}$. In other words, the $\underline{\underline{3N}}_{H\#}$ arithmetic -- that is, that human subject, as this arithmetic, 'mentally incarnating' this arithmetical axioms-system -- thus finds itself wanting, due to this internally needed but -- explicitly, at least -- internally lacking content. It finds itself lacking, 'self-wanting', wanting its own fuller self, its own more "'complete'" self, its own fuller becoming -- wanting its own fuller 'self-becoming', in the form of a system of arithmetic which explicitly embraces, 'taxonomizes', and explains more of that totality of the modern, contemporary "Standard Arithmetics" that we all have experienced, even if only "chaotically" [Marx] so far.

The ' $\underline{\underline{3N}}_{H\#}$ ' side of ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ ' 'explicitizes' only the 'objectifiedly'-'held-on-paper' expression/encoding pole of the meaning of $\underline{\underline{3N}}_{H\#}$ -- as one of the two poles of $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ as a whole -- alone, by itself, hence signifies only the "object" pole of the ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ ' whole of step/stage 1.

The ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{\quad}})$ ' side of ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ ' 'explicitizes' only the 'subjectifiedly'-'held-in-mind' expression/encoding pole of the meaning of $\underline{\underline{3N}}_{H\#}$ -- as the other of the two poles of $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ as a whole -- alone, by itself, hence signifies only the "subject" pole of the ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ ' whole of step/stage 1.

The ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ ' whole of step/stage 1 signifies the mutual "clash" of these two, opposite poles, or the 'self-clash' of $\underline{\underline{3N}}_{H\#}$ within itself; the 'explicitization' / - externalization of the immanent 'self-clash', 'self-antithesis', or 'self-duality' of $\underline{\underline{3N}}_{H\#}$, which is, namely, 'counts-as-numbers versus non-counts-as-numbers'.

Thus, ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{3N}}_{H\#})$ ' connotes our state of consciousness -- the state of consciousness of we who are following the $\underline{\underline{3N}}_{H\#} \stackrel{\#}{=} \underline{\underline{3N}}_{H\#}^{[0,6]}$ Seldon Function "'recipe'" for this 'self-dialogue', for $s_{\#} = 1$ -- wherein ' $\underline{\underline{3N}}_{H\#}(\underline{\underline{\quad}})$ ', as connoting the "'spirit'" of the meaning of $\underline{\underline{3N}}_{H\#}$, confronts ' $\underline{\underline{3N}}_{H\#}$ ' as connoting the "'letter'" of the meaning of $\underline{\underline{3N}}_{H\#}$.

This interaction of $\underline{\underline{3N}}_{H\#}$ as "'subject'" with $\underline{\underline{3N}}_{H\#}$ as "'object'" describes the state of activity of mind which produces the result of step/stage 1.

This “‘dialectical ideography’” is thus a “‘shorthand’”, describing what our minds are doing -- or should be doing -- so as to follow the $\frac{3}{H} \underline{H}^{\#}_6 = \left(\frac{3}{H} \underline{N}^{\#} \right)^2_6$ dialectical method of presentation.

The principles delineated above for the special case of Step/Stage 1 also hold in general throughout this presentation, for each subsequent Step/Stage, but will not be repeated in detail in each later presentational locus for which they hold.

Taking the Measure of Where We Have Arrived Cognitively via Step/Stage 1 in this Chain of Immanent Critiques.

Thus, we arrive at -- as the fruition of Step/Stage 1 -- the ‘mutual controversion’ of $\frac{3}{H} \underline{N}^{\#}$ versus $\frac{3}{H} \underline{a}^{\#}$.

We have $\frac{3}{H} \underline{N}^{\#}$ as the partial, one-sided, unnecessarily restrictive view of modern ‘Standard Number’ as limited to “‘counting number’”, to “‘cardinal number’”, to “‘cardinality’”.

And we have that $\frac{3}{H} \underline{N}^{\#}$ in relation to $\frac{3}{H} \underline{a}^{\#}$ as the *counter-example* to $\frac{3}{H} \underline{N}^{\#}$ ’s “‘number as cardinality’” [and *only* as “‘cardinality’”] view. That is, we have $\frac{3}{H} \underline{a}^{\#}$ as a [sub-]system of “‘non-cardinal’” number; of “‘non-cardinality’”.

However, in itself, $\frac{3}{H} \underline{a}^{\#}$ is just as “one-sided” and “partial” -- just as incomplete and inadequate -- as $\frac{3}{H} \underline{N}^{\#}$, if not even more so.

The $\frac{3}{H} \underline{a}^{\#}$ [sub-]system of arithmetic represents *only* “‘non-cardinality’”, so that $\frac{3}{H} \underline{N}^{\#}$ is equally a *counter-example* to $\frac{3}{H} \underline{a}^{\#}$, or to any claims of *adequacy*, *completeness*, and *totality* that might be ‘mis-made’ on behalf of $\frac{3}{H} \underline{a}^{\#}$.

We thus remain in an unsatisfactory situation, in terms of systematically -- including of “‘taxonomically’” -- organizing and accounting for our entire experience of the totality of modern, contemporary “Standard Arithmetic(s)”, although the ‘explicitization’ of $\frac{3}{H} \underline{a}^{\#}$ does mark an *advance* over the *even more unsatisfactory situation* of Step/Stage 0.

But to *improve further our situation*, we must move on, on to Step/Stage 2, to $\frac{3}{H} \underline{H}^{\#}_2 = \left(\frac{3}{H} \underline{N}^{\#} \right)^2_2$, i.e., to the self-reflection, or “‘squaring with itself’”, of Step/Stage 1’s own result, namely, of $\left(\frac{3}{H} \underline{N}^{\#} \oplus \frac{3}{H} \underline{a}^{\#} \right)$ [with] itself.

C.2.2. stage/step S = 2: Consolidation and ‘self-aporization’ of the System of Arithmetic of the “**Whole Numbers**”, **W**.

- 2.1. descriptive name of Stage: *The Consolidation of the “Whole numbers” and their “‘Formal Subsumption’” by the “minus numbers”*.
- 2.2. stage ‘parametrics’: [total terms #, 2^2] **4**; [new terms #, 2^{2-1}] **2**; [new terms needing solution #, $(2^{2-1} - 2)$] **0**.
- 2.3. «*aporia*» of this stage: $\overset{3}{H}\overset{W}{\#} \dashv \oplus \dashv \overset{3}{H}\overset{m}{\#}$.
- 2.4. “‘incompleteness’”-revealing “unsolvable” “diophantine” equations-family for this stage: $W + x_2 = 0$; x_2 & $W \neq 0$, $W \in W$.
- 2.5. “unsolvable” equations’ paradox: *The Paradox of ‘Decreasive’ Addition* [i.e., of ‘*Subtractive Addition*’] *within* $\overset{3}{H}\overset{a}{\#}$.
- 2.6. shortcut rendition of the ‘product-tion’ of Stage 2 from Stage 1 [via the ‘meta-meristematic principle’]:

$$\begin{aligned}
{}^3\text{H}^{\#}_{\underline{\#}_2} &= {}^3\text{H}^{\#}_{\underline{\#}} \otimes {}^3\text{H}^{\#}_{\underline{\#}_1} = {}^3\text{H}^{\#}_{\underline{\#}} \otimes ({}^3\text{N}^{\#}_{\underline{\#}} \oplus {}^3\text{a}^{\#}_{\underline{\#}}) = \\
({}^3\text{H}^{\#}_{\underline{\#}} \otimes {}^3\text{N}^{\#}_{\underline{\#}}) \oplus ({}^3\text{H}^{\#}_{\underline{\#}} \otimes {}^3\text{a}^{\#}_{\underline{\#}}) &= {}^3\text{N}^{\#}_{\underline{\#}} \oplus {}^3\text{a}^{\#}_{\underline{\#}} \oplus {}^3\text{H}^{\#}_{\underline{\#}} \otimes \text{aN}^{\#} \oplus {}^3\text{H}^{\#}_{\underline{\#}} \equiv \\
{}^3\text{W}^{\#}_{\underline{\#}} \sqcup \oplus {}^3\text{H}^{\#}_{\underline{\#}} \text{m}^{\#}
\end{aligned}$$

- 2.7.** 'pre-solved' new terms -- new terms of *given* or *already solved* meanings/definitions:

2.7.1. $\equiv \left(\left(\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{W} \right) \right) \vdash \vdash_3$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

number-space that expresses the fruition of the critique of $\overset{3}{\text{H}}\text{N}_{\#}$ by $\overset{3}{\text{H}}\text{a}_{\#}$ in the product $\overset{3}{\text{H}}\text{a}_{\#} \otimes \overset{3}{\text{H}}\text{N}_{\#}$, namely $\overset{3}{\text{H}}\text{a}_{\#} \otimes \overset{3}{\text{H}}\text{N}_{\#} \equiv \overset{3}{\text{H}}\text{w}_{\#}$.

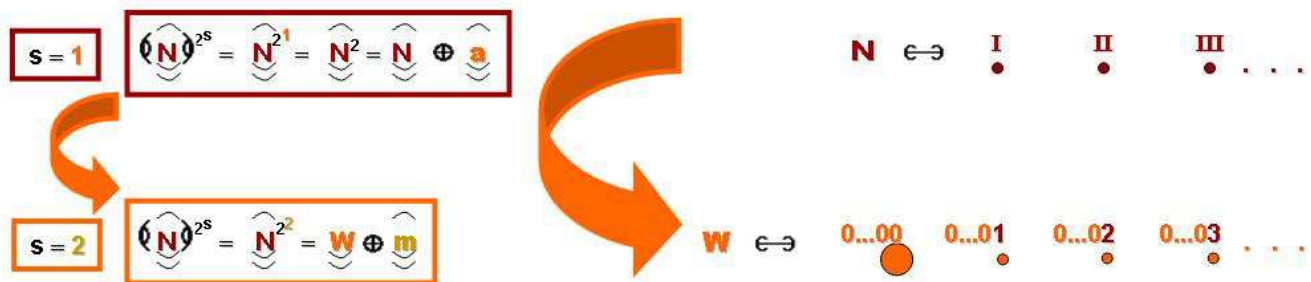
the sub-space of the **W** number-space containing all $\overset{3}{\text{H}}\text{a}_{\#}$ -converted “**N**atural” Numbers, $\mathbf{N} = \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \dots\}$.

2.7.2. $\equiv \left(\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{m}_{\#} \right) \text{E} \rightarrow \begin{smallmatrix} \text{I} \\ 4 \end{smallmatrix}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the arithmetic of the “minus” numbers, $\text{m} \equiv \{0-\text{I}, 0-\text{II}, 0-\text{III}, \dots\}$, the sub-space of the Z number-space containing all “negative” differences between the “Natural” Numbers, $\text{N} \equiv \{\text{I}, \text{II}, \text{III}, \dots\}$, and the “aught” number 0 ; fruition of the [immanent] self-critique of $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{a}_{\#}$, namely $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{aa}_{\#} \equiv \begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{m}_{\#}$. Because $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{a}_{\#}$ is already ingredient in $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{m}_{\#} \equiv \begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{aa}_{\#}$, $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{m}_{\#}$ is already ‘pre-aught-ized’, or ‘zero-ized’/‘0-ized’, by its inherence/inheritance/ internal conservation of $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \text{a}_{\#}$, i.e., in full, $\text{m} \equiv \{0-\text{I} \equiv -0...01, 0-\text{II} \equiv -0...02, \dots\}$.

- 2.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module ζ.]:** \neq .

Number-Line Cumulative Progression Analytic-“Geometric” Model
of the **Gödelian Dialectic of the Standard Arithmetics**:

Transition from $s = 1$ to $s = 2$, and from $\mathbf{N} \oplus \dots$ to $\mathbf{W} \oplus \dots$



Narrative Commentary for stage S = 2. ¿An Opposite for the “Whole” Numbers **W**? The stage 2 ‘ideo-cumulum’ of our ‘meta-model’ ‘super[im]poses’ 4 terms upon one another, the first 3 of which are $\frac{3}{H} \frac{N}{\#} \oplus \frac{3}{H} \frac{a}{\#} \oplus \frac{3}{H} \frac{a}{\#} \frac{N}{\#}$, which, together, we find, connote the “Whole” Numbers system of arithmetic, $\frac{3}{H} \frac{W}{\#}$.

We so find, because the 1st term, $\frac{3}{H} \frac{N}{\#}$, connotes that universal base/core that is the “Natural” Numbers system, the 2nd term, $\frac{3}{H} \frac{a}{\#}$, connotes ‘the zeros-sub-system’, and the 3rd term, $\frac{3}{H} \frac{a}{\#} \frac{N}{\#}$, connotes the complex unity of/“real subsumption” of that core “Natural” system, $\frac{3}{H} \frac{N}{\#}$, with/by ‘the zeros-sub-system’, $\frac{3}{H} \frac{a}{\#}$, thus combining the forerunner base system of the **N**, the ‘contra-system’ of the **a**, & the implementation of **a** upon **N**, all together connoting **W** $\equiv \{0, 1, 2, 3, \dots\}$.

These first three terms, taken together as $\frac{3}{H} \frac{W}{\#}$, form a new «*aporia*» with the 4th and final term for stage 2: $\Delta(\frac{3}{H} \frac{a}{\#}) = \frac{3}{H} \frac{a}{\#} \frac{aa}{\#}$. ¿What, then, *should* this

4th term mean? ¿Is there a kind of arithmetic, that we have encountered in our, perhaps “chaotic”, experience of the totality of the modern/contemporary “Standard Arithmetics”, based upon a kind of number(s), and an arithmetical ‘ideo-phenomenon’ supported by that kind of number(s), which is opposite, in kind, in quality, to the ‘ideo-phenomena’ of the “Whole” Numbers *as a whole* -- not just to those of some specific, individual “Whole” number(s), but to the whole “genre” of number(s)

“Whole”-ness? ¿More specifically, is there a kind of number(s) *opposite* [‘ \leftarrow ’] to the $\frac{3}{H} \frac{a}{\#}$ kind of number(s), that corresponds to the fruition of the *self-critique* of $\frac{3}{H} \frac{a}{\#}$, namely, to $\frac{3}{H} \frac{a}{\#} \frac{aa}{\#} \equiv \Delta(\frac{3}{H} \frac{a}{\#})$?

The ‘a-ization’ of the **N**, the addition, to the incipient number “line”, or number “ray”, in the ‘sub-diagram’ [in Background section **B.η** of Part **I**.] depicting the **N**, of the “point” representing the *zero(s)*, the “origin”, to the immediate left of the “point” representing the “Natural” number **I**, to form the “Whole” Numbers’ ‘number-ray’ in the next ‘sub-diagram’ of that figure, has worked a subtle transformation in point-of-view. It has heightened a sense of ‘directionality’ already latent in the “Natural” Numbers, by contrasting the implicit “uni-directionality” of the “Natural” Numbers, which mount in magnitude to the right, with the apparent presence, for “Whole” number “analytical geometry”, on “their” left, of a new “point”, *apparently directionless*, representing the **0s**. Noticing this contrast of “directionless” **0** & implicitly *one-directional N* within itself, $\frac{3}{H} \frac{a}{\#} \equiv \frac{3}{H} \frac{a}{\#} \frac{NN}{\#}$, immanently critiquing itself, $\frac{3}{H} \frac{a}{\#} \otimes \frac{3}{H} \frac{a}{\#} = \frac{3}{H} \frac{a}{\#} \frac{aa}{\#} \oplus \frac{3}{H} \frac{a}{\#} \frac{aa}{\#} = \frac{3}{H} \frac{a}{\#} \frac{aa}{\#} \frac{aa}{\#}$, issues

forth with a full ‘explicitization’ of [co-linear] ‘counting number directionality’, positing the possibility of counting “in the opposite direction” from the only direction of mounting counting that is extant in **N**. For example, two units of currency, *counted in the debit* “direction”, can cancel -- to **0** -- two units of currency, *counted in the credit* “direction”. It is important to notice here that the **m**, the numbers of strictly ‘negative directionality’, are *opposite* to, not just the numbers of strictly ‘positive directionality’, that are latent, as such, in **N**, but are *opposite* to the *totality* of the **W** -- are *contrary* to the whole combination of ‘uni-directional’ and ‘direction-less’ numbers that found and ground $\frac{3}{H} \frac{W}{\#}$: $\{0 - I \equiv -0...01, 0 - II \equiv -0...02, 0 - III \equiv -0...03, \dots\} \equiv \{\dots, -3, -2, -1\} \equiv m \leftarrow W$.

Strictly in terms of our “analytic-geometric” vision/visualization of the Standard Arithmetics, in **B.η**, **0** filled in a slot to the left of all of the “Natural” Numbers: **0** is therefore “less than” any/all of the “Natural” Numbers. The fruition of the *self-critique* of $\frac{3}{H} \frac{a}{\#}$ is “the further filling-in of [co-linear] number-space, to the left of **0**”

in consolidating the space of **W**, of the “Whole” Numbers system, $\frac{3}{H} \frac{W}{\#}$. That fruition, $\frac{3}{H} \frac{m}{\#}$, marks/names a “filling-in” of a potentially-infinite ‘counter-ray’ of

[co-linear] number-space “points”, for number-values that mount in *absolute*-value magnitude in a direction *opposite* to that in which the **N** mount in [absolute[-value]] magnitude[, & contrary also to the **0s**, which do not mount in *any* direction], & thus represent number-values that are “less than zero” -- $\zeta \equiv$ “less than nothing” ? Well, one key to the way out of this paradox, of this «*aporia*», is the realization that *zero* does not exactly stand for “nothing” -- for “absolute/total nothingness” -- but only for the *absence* of any *counts* of a[n implicitly qualitative] *unit* [i.e., of one] *of a particular kind* in a given context of arithmetical modeling of sensuous actuality, so that a value of **0** may only mean that no *unit of the given kind* is present in the implicitly given context/locus of discourse, even while everything else -- even of all *units of every other kind* or «*genos*»-- might even be “all present and accounted for”; the rest of the universe[-of-discourse] thus still perfectly intact, despite the apparent local *disappearance* of all *units of the particular kind in question*. Because **0** is a “pure *quantifier*”, without any explicit “ideographical”, ‘numeralic’, arithmetical, mathematical *kind-qualifier* -- because ‘Standard Numbers’ through at least the **R** ‘elide’ all *explicit* such “qualification” -- the mere Indo-Arabic ‘numeralic’ value **0** is ambiguous as to what has become *absent*, and as to what has *not*.

The Equation, “‘Well-Formed’” $\text{within } \frac{3}{H} \underline{W}$, that is *Not Solvable* $\text{within } \frac{3}{H} \underline{W}$, and that Therefore Points, from $\text{within } \frac{3}{H} \underline{W}$, to *beyond* $\frac{3}{H} \underline{W}$ -- to $\text{within } \frac{3}{H} \underline{Z}$.

In terms of an immanent critique, or self-critique, of the “Whole” Numbers system of arithmetic, and, more specifically, in terms of the self-critique of the “aught” numbers sub-system of arithmetic, we have noted a family of algebraic equations, “‘well-formed’” within the algebra of the “Whole” Numbers’ arithmetic, which are, however, not “satisfiable” within the “Whole” Numbers system -- which are not “solvable” by any “Whole” number(s). This equations-family can be represented, generically, by the single equation, $[W + x_\beta = 0]$, via the ‘variable constant’ [via the “parameter”] $W \mid W \in W$. This equation represents a *paradox* from the point of view of a definition of ‘Standard Number’ restricted within the concept of the “Whole” Numbers. The “Whole” Numbers concept can support the addition of an unknown number to a known “Whole” number that leaves that known “Whole” number unchanged, neither increased or decreased: $W + x_\alpha = W$. The solution to that equation, as reviewed in the just-previous stage of our ‘self-argument’, is, of course, ‘the aught number(s)’, $0: x_\alpha = 0$, the “additive identity element”. But the “Whole” Numbers cannot supply, for every individual “Whole” number, W , an ‘anti-[Whole]-number’ -- another “Whole” number, which, when added to the non-zero “Whole” number in question, W , “mutually annihilates” with that W ; reduces that non-zero “Whole” number [and itself] back to “neutral” -- back to 0 . Such ‘*decreasing addition*’, or ‘*subtractive addition*’, is beyond the ken of the “Whole” Numbers *«mentalité»*.

Therefore, the self-critique of the “Whole” Numbers, as seeded by the above-described equation, &, more specifically, by the self-critique of ‘the aught number(s)’, leads us to the “minus numbers”, $\frac{3}{H} \underline{aa} \equiv \frac{3}{H} \underline{m}$, the numbers which are both ‘left of zero’ [“‘analytic-geometrically’”], & “less than zero”, but *not* “less than

nothing”. The \underline{m} , in this stage’s now “‘formal subsumption’” of/contrast of the \underline{m} & their [sub-]system of arithmetic, $\frac{3}{H} \underline{m}$, vs. the W , & their system of arithmetic,

$\frac{3}{H} \underline{W}$, & in terms of our “‘analytic geometric’” visualization of these “number-spaces”, redefine 0 to be, *not* a numeral naming “nothing”, but the *third thing*, ‘the neutral position’, the *«tertium quid»*, lodged between, and *dividing*, the “first thing” of counting/absolute-value-mounting in the *plus/rightward direction* -- “positive counting”, e.g., *credit* [ac]counting -- from the “second thing”, counting/absolute-value-mounting in the *minus/leftward direction* -- “negative counting”, e.g., *debit* [ac]counting.

This stage of our ‘self-dialogue’ has therefore further “filled-in” our “‘analytic-geometric’” number-space. It has equipped us conceptually with a whole, new, potentially-infinite ‘number-ray’ of an incipient “‘number-line’”, a new ray which points/veers off to the left, complementing and supplementing the original/earlier number-spaces of the N and the W . The latter are now ‘re-seen’, in this new relative light, as potentially -- and as explicitly, in retrospect -- an oppositely-oriented number [ar]ray, pointing/veering off to the right. Thus, both are, together, [re-]visualized as mutually-oppositely-oriented number [ar]rays, or number ‘sub-lines’, which potentially inhere in a single, co-linear, rectilinear -- but ‘*bi-directional*’ -- “dimension” of number: potentially a single “‘number-line’”, but one *divided* *within*, as a two-sided, but also as a ‘*tri-componented*’ $\{-, \pm, +\}$, rectilinear “number-space”.

Taking the Measure of Where We Have Arrived Cognitively via Step/Stage 2 in this Chain of Immanent Critiques.

Thus, we arrive at -- as the fruition of Step/Stage 2 -- the ‘mutual controversion’ of $\frac{3}{H} \underline{W}$ vs. $\frac{3}{H} \underline{m}$. We have $\frac{3}{H} \underline{W}$ as the partial, one-sided, unnecessarily restrictive view of modern ‘Standard Number’ as limited to either “‘non-counting’” or to ‘rightward-counting/-mounting from a no-count origin/starting-point’. And we have this $\frac{3}{H} \underline{W}$, in [oppositional-addition-]relation to $\frac{3}{H} \underline{m}$ as the *counter-example* to $\frac{3}{H} \underline{W}$ ’s ‘number as direction-less non-counting’ or ‘as uni-directional counting’, view. That is, we have $\frac{3}{H} \underline{m}$ as a [sub-]system of ‘leftward-counting/[absolute-value-]mounting from a no-count origin/starting-point’. But, in itself, $\frac{3}{H} \underline{m}$ is just as “one-sided”/“partial” -- just as incomplete/inadequate -- as $\frac{3}{H} \underline{W}$, if not even more so. The $\frac{3}{H} \underline{m}$ [sub-]system of arithmetic encompasses *only* “‘leftward absolute-value mounting’”, excluding even the neutral point, 0 . So $\frac{3}{H} \underline{W}$ is equally a *counter-example* to $\frac{3}{H} \underline{m}$ -- to any claims of adequacy/completeness/totality that might be made on behalf of $\frac{3}{H} \underline{m}$.

We thus remain in an unsatisfactory situation, in terms of systematically -- including of “‘taxonomically’” -- organizing & accounting for our entire experience of the totality of modern, contemporary “Standard Arithmetic(s)”, although the ‘explicitization’ of $\frac{3}{H} \underline{m}$ does mark an advance over the even more unsatisfactory situation

wherein we dwelt in Step/Stage 1. To improve our situation further, we must move on, on to Step/Stage 3, to $\frac{3}{H} \underline{H} = \left(\frac{3}{H} \underline{N} \right)^2$, i.e., to the *self-reflection*,

or “‘squaring with itself’”, of Step/Stage 2’s own whole result, namely, of $\left(\frac{3}{H} \underline{N} \oplus \frac{3}{H} \underline{a} \oplus \frac{3}{H} \underline{aa} \oplus \frac{3}{H} \underline{m} \right)$ [with] itself, or [shortcut] of the operation of the *«aufheben»* operation/operator $\frac{3}{H} \underline{m}$ upon that whole Step/Stage 2 result -- the ‘[de]flection’ [“‘bending’”] of that whole result by $\frac{3}{H} \underline{m}$.

What the Symbol $\overset{3}{\underset{h}{W}}$ “Intends”: *Axioms-System of the Arithmetic of the so-called “Whole” Numbers, \mathbf{W}* [commenced]

W0. \mathbf{W} is not the Empty Set.

$\mathbf{W} \neq \emptyset \equiv \{ \};$

First Order Peano Postulates: [mainly] “Phonogramic” Rendering

[mainly] “Ideogramic” Rendering

WP1. $\mathbf{0}$ is a “Whole” Number. [$\subset \Delta \mathbf{N}$]

$\mathbf{0} \in \mathbf{W};$

WP2. The successor of a “Whole” Number is also a “Whole” Number.

$w \in \mathbf{W} \Rightarrow s(w) \in \mathbf{W};$

WP3. No two “Whole” Numbers have the same successor.

$w, m \in \mathbf{W} \text{ \& } w \neq m \Rightarrow s(w) \neq s(m);$

WP4. $\mathbf{0}$ is not the successor of any “Whole” Number. [$\subset \Delta \mathbf{N}$]

$\neg \exists x \in \mathbf{W} \mid s(x) = \mathbf{0};$

Second Order Peano Generalized *Mathematical Induction* Postulate, for “Whole” Number Arithmetic, for Predicate P

WP5. $[\forall P][\forall w_1, w \in \mathbf{W}][[P(w_1)] \text{ \& } [P(w_1) \Rightarrow [P(s(w_1))]]] \Rightarrow [[\forall w \in \mathbf{W}][[w \geq w_1][P(w)]]];$

First Order Axioms of “Whole” Number **Addition**:

WA1. For all w_1, w_2 in \mathbf{W} , $w_1 + w_2 = w_2 + w_1$ [Additive Commutativity];

WA2. For all w_1, w_2, w_3 in \mathbf{W} , $(w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$ [Additive Associativity];

WA3. For all w_1, w_2 in \mathbf{W} , $w_1 + w_2$ is in \mathbf{W} [Additive Closure of \mathbf{W}]; $[\forall w_1, w_2 \in \mathbf{W}][w_1 + w_2 \in \mathbf{W}];$

WA4. There is an element $\mathbf{0}$ in \mathbf{W} such that, for all w in \mathbf{W} , $\mathbf{0} + w = w$ [Additive Invariance Element] [$\subset \Delta \mathbf{N}$];

$[\exists \mathbf{0} \in \mathbf{W}][[\forall w \in \mathbf{W}][\mathbf{0} + w = w + \mathbf{0} = w];$

First Order Axioms of “Whole” Number **Multiplication**:

WM0. There is an element $\mathbf{0}$ in \mathbf{W} such that, for all w in \mathbf{W} , $\mathbf{0} \times w = \mathbf{0}$ [Multiplicative Predominance] [$\subset \Delta \mathbf{N}$];

$[\exists \mathbf{0} \in \mathbf{W}][[\forall w \in \mathbf{W}][\mathbf{0} \times w = w \times \mathbf{0} = \mathbf{0}];$

WM1. For all w_1, w_2 in \mathbf{W} , $w_1 \times w_2 = w_2 \times w_1$ [Multiplicative Commutativity];

WM2. For all w_1, w_2, w_3 in \mathbf{W} , $(w_1 \times w_2) \times w_3 = w_1 \times (w_2 \times w_3)$ [Multiplicative Associativity];

WM3. For all w_1, w_2 in \mathbf{W} , $w_1 \times w_2$ is in \mathbf{W} [Multiplicative Closure of \mathbf{W}]; $[\forall w_1, w_2 \in \mathbf{W}][w_1 \times w_2 \in \mathbf{W}];$

WM4. There is an element $\mathbf{1}$ in \mathbf{W} such that, for all w in \mathbf{W} , $\mathbf{1} \times w = w$ [Multiplicative Invariance Element];

$[\exists \mathbf{1} \in \mathbf{W}][[\forall w \in \mathbf{W}][\mathbf{1} \times w = w \times \mathbf{1} = w];$

What the Symbol $\overset{3}{\underset{h}{W}}$ “Intends” [continued & concluded].

First Order Axioms for the “Hybridization” of the “Whole” Number Operations of “Multiplication” and of “Addition”:

WH1. For all w_1, w_2, w_3 in \mathbf{W} , $(w_1 + w_2) \times w_3 = w_1 \times w_3 + w_2 \times w_3;$

$[\forall w_1, w_2, w_3 \in \mathbf{W}][[(w_1 + w_2) \times w_3 = w_1 \times w_3 + w_2 \times w_3]]$ [“Distributivity” [“Hybridization”] of Addition over Multiplication];

WH2. For all w_1, w_2, w_3 in \mathbf{W} , $w_1 \times (w_2 + w_3) = w_1 \times w_2 + w_1 \times w_3;$

$[\forall w_1, w_2, w_3 \in \mathbf{W}][[w_1 \times (w_2 + w_3) = w_1 \times w_2 + w_1 \times w_3]]$ [“Distributivity” [“Hybridization”] of Multiplication over Addition];

First Order Axioms of “Whole” Number **Exponentiation**:

WE0. For all w in $\mathbf{W} - \{\mathbf{0}\}$, $w^0 = \mathbf{1}$; the value $\mathbf{0}^0$ is undefined [Exponentiation Predominance] [$\subset \Delta \mathbf{N}$];

WE1. For all w_1, w_2, w_3 in \mathbf{W} , $w_1^{w_2} \times w_1^{w_3} = w_1^{w_2 + w_3}$ [Exponentiation Additivity];

WE2. For all w_1, w_2, w_3 in \mathbf{W} , $(w_1^{w_2})^{w_3} = w_1^{w_2 \times w_3}$ [Exponentiation Multiplicativity];

WE3. For all w_1, w_2, w_3 in \mathbf{W} , $(w_1 \times w_2)^{w_3} = w_1^{w_3} \times w_2^{w_3}$ [Exponentiation “Distributivity”];

~~**WE4.** For all w_1, w_2 in \mathbf{W} , $w_1^{w_2}$ is in \mathbf{W} [Exponentiation Closure] [$\subset \Delta \mathbf{N}$]; $\mathbf{0}^0 \notin \mathbf{W}$;~~

WE5. For all w in \mathbf{W} , $w^1 = w$ [Exponentiation Invariance];

Axioms of “Whole” Number **Order**:

WO1. For all w_1, w_2 in \mathbf{W} , either $w_1 > w_2$, or $w_1 = w_2$, or $w_1 < w_2$ [Trichotomy “Law” [First Order]];

$[\forall w_1, w_2 \in \mathbf{W}][[w_1 > w_2] \vee [w_1 = w_2] \vee [w_1 < w_2]];$

WO2. For all $w_1 \neq \mathbf{0}, w_2 \neq \mathbf{0}$ in \mathbf{W} , $w_1 + w_2 > w_1, w_2$ [Additive Order [First Order]] [$\subset \Delta \mathbf{N}$];

$[\forall w_1, w_2 \in \mathbf{W} \mid w_1 \neq \mathbf{0} \neq w_2][w_1 + w_2 > w_1, w_2];$

WO3. For all $w_1 \neq \mathbf{0}, w_2 \neq \mathbf{0}$ in \mathbf{W} , $w_1 \times w_2 > w_1, w_2$ [Multiplicative Order [First Order]] [$\subset \Delta \mathbf{N}$];

$[\forall w_1, w_2 \in \mathbf{W}][[w_1 \neq \mathbf{0} \neq w_2] \Rightarrow [w_1 \times w_2 > w_1, w_2]] \text{ \& } [[w_1 \times w_2 = \mathbf{0}] \Leftrightarrow [[w_1 = \mathbf{0}] \vee [w_2 = \mathbf{0}]]];$

WO4. Any non-empty set of “Whole” Numbers contains a least element [Well Ordering Principle [Second Order]];

$[\forall S \subseteq \mathbf{W}][[S \neq \emptyset][\exists s \in S][\forall w \in S][w \leq s];$

WO5. $\mathbf{0} < \mathbf{1}$ [Order of Invariance Elements [First Order]] [$\subset \Delta \mathbf{N}$].

C.2.3. stage/step S = 3: Consolidation and ‘self-aporization’ of the System of Arithmetics of the “Integers”, **Z**.

3.1. descriptive name of stage: *The Consolidation of the “Integers” and their “‘Formal Subsumption’” by the “fractional numbers”*.

3.2. stage ‘parametrics’: [total terms count, 2^3] **8**; [new terms count, 2^{3-1}] **4**; [new terms needing solution count, $(2^{3-1} - 2)$] **2**.

3.3. «aporia» of this stage: $\frac{3}{H}Z \vdash \oplus \frac{3}{H}f$.

3.4. “‘incompleteness’”-revealing “unsolvable” inequations-family for this stage: $|x_3 \times z| < |z|, |x_3| > \pm 0, z \in Z$.

3.5. “unsolvable” inequations’ paradox: *The Paradox of ‘Decreasive’ Multiplication within $\frac{3}{H}m$* .

3.6. shortcut rendition of the ‘product-tion’ of stage 3 from stage 2 [via the ‘meta-meristematic principle’]:

$$\begin{aligned} \frac{3}{H}H_3^{\#} &= \frac{3}{H}m \otimes \frac{3}{H}H_2^{\#} = \frac{3}{H}m \otimes \left(\frac{3}{H}N \oplus \frac{3}{H}a \oplus \frac{3}{H}a_N^{\#} \oplus \frac{3}{H}m \right) = \\ & \left(\frac{3}{H}m \otimes \frac{3}{H}N \right) \oplus \left(\frac{3}{H}m \otimes \frac{3}{H}a \right) \oplus \left(\frac{3}{H}m \otimes \frac{3}{H}a_N^{\#} \right) \oplus \left(\frac{3}{H}m \otimes \frac{3}{H}m \right) = \\ & \frac{3}{H}Z \vdash \oplus \frac{3}{H}f. \end{aligned}$$

3.7. ‘pre-solved’ new terms -- new terms of *given* or *already solved* meanings/definitions:

3.7.1. $\equiv \left(\frac{3}{H}z \right) \vdash \mathbb{Q}_7$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’

of the “‘epitome’” of the *Integers*, $z \equiv \{+0...01, +0...02, +0...03, \dots\}$, the sub-space of the **Z** number-space which expresses the

fruition of the *critique* of $\frac{3}{H}a_N^{\#}$ by $\frac{3}{H}m$ in the product $\frac{3}{H}m \otimes \frac{3}{H}a_N^{\#} = \frac{3}{H}m \otimes \frac{3}{H}w$, i.e., $\frac{3}{H}a_N^{\#} = \frac{3}{H}z$.

the sub-space of the **Z** number-space containing all $\frac{3}{H}m$ -converted $\frac{3}{H}a$ -converted “Natural” Numbers, originally $N \equiv \{I, II, III, \dots\}$.

3.7.2. $\equiv \left(\frac{3}{H}f \right) \vdash \mathbb{Q}_8$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’

of the “‘fractional’” numbers, $f \equiv \{(z_1/z_2) | z_1, z_2 \in m \subset Z\} \equiv Q_+ \subset Q$, the sub-space of the **Q** number-space containing all

Quotients of “minus” numbers, $m \equiv \{0-I, 0-II, 0-III, \dots\}$, ‘pre-excluding’ ± 0 as a denominator, since $\pm 0 \notin m$; fruition of the

[immanent] ‘self-critique’ of $\frac{3}{H}m$, namely, $\frac{3}{H}a_{mm}^{\#} = \frac{3}{H}f$.

3.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module ζ_1]:

3.8.1. $\equiv \left(\frac{3}{H}a_{mN}^{\#} \right) \vdash \mathbb{Q}_5$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘*integers-ized*’ “Natural” Numbers, $mN \equiv \{+I, +II, +III, \dots\}$, the sub-space of the **Z** number-space which

expresses the fruition of the *critique* of $\frac{3}{H}N$ by $\frac{3}{H}m$ in the product $\frac{3}{H}m \otimes \frac{3}{H}N$, namely $\frac{3}{H}a_{mN}^{\#}$.

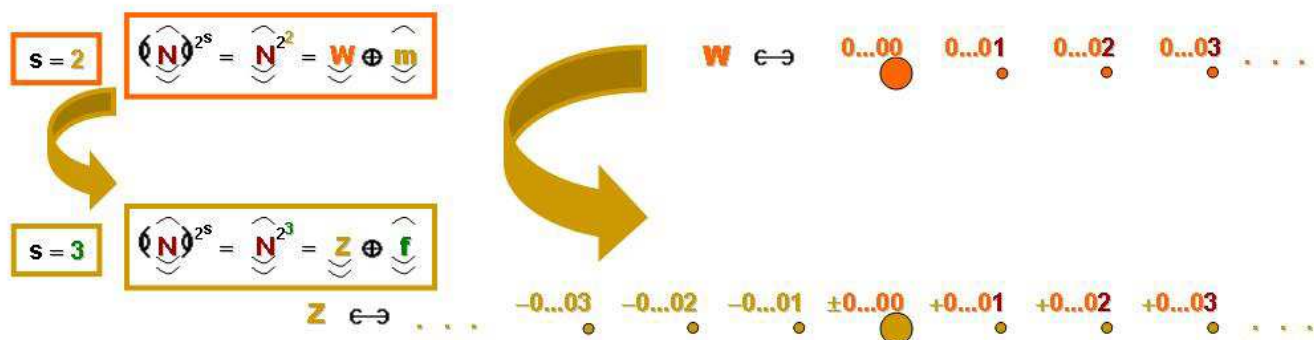
3.8.2. $\equiv \left(\frac{3}{H}a_{ma}^{\#} \right) \vdash \mathbb{Q}_6$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘*integers-ized*’ ‘aught’ Numbers, $ma \equiv \{I-I=\pm 0, II-II=\pm 0, III-III=\pm 0, \dots\}$, the sub-space of the **Z**

number-space which expresses the fruition of the *critique* of $\frac{3}{H}a$ by $\frac{3}{H}m$ in the product $\frac{3}{H}m \otimes \frac{3}{H}a$, namely $\frac{3}{H}a_{ma}^{\#}$.

Number-Line Cumulative Progression Analytic-“Geometric” Model
of the **Gödelian Dialectic of the Standard Arithmetics**:

The Transition from $s = 2$ to $s = 3$, and from $W \oplus \dots$ to $Z \oplus \dots$



Narrative Commentary for stage $s = 3$. ¿An Opposite for the “*Integers*”, \mathbf{Z} ? The stage 3 ‘ideo-cumulum’ of our ‘meta-model’ ‘super[im]poses’ 8 terms upon one another, the first 7 of which are $\frac{3}{H}N_{\#} \oplus \frac{3}{H}a_{\#} \oplus \frac{3}{H}a_{\#}N \oplus \frac{3}{H}m_{\#} \oplus \frac{3}{H}m_{\#}N \oplus \frac{3}{H}ma_{\#} \oplus \frac{3}{H}ma_{\#}N$, which, together, we

find, connote the “*Integers*” system of arithmetic, $\frac{3}{H}Z_{\#}$. We so find, because the first three terms connote the “*Whole*” Numbers system of arithmetic, $\frac{3}{H}W_{\#}$, which is “contained in” the “*Integers*” system of arithmetic, $\frac{3}{H}Z_{\#}$, and such that $W \equiv \{0, 1, 2, 3, \dots\}$, whereas $Z \equiv \{\dots, -3, -2, -1, \pm 0, +1, +2, +3, \dots\}$.

These first seven terms, taken together as connoting $\frac{3}{H}Z_{\#}$, form a new «*aporia*» with the 8th and final term for stage 3: $\Delta(\frac{3}{H}m_{\#}) \equiv \frac{3}{H}mm_{\#}$. ¿What, then,

should this 8th term mean? ¿Is there a kind of arithmetic, that we have encountered in our, perhaps “chaotic”, experience of the totality of the modern/contemporary “Standard Arithmetics”, based upon a kind of number(s), and an arithmetical ‘ideo-phenomenon’ supported by that kind of number(s), which is *opposite*, in kind, in quality, to the ‘ideo-phenomena’ of the “*Integers*” system *as a whole* -- not just to those of some specific, individual “*Integer*”, but to the whole “*genre*” of number

‘*Integ[er]ity*’? ¿More specifically, is there a kind of number(s) that is *opposite* to the $\frac{3}{H}m_{\#}$ kind of number(s), that corresponds to the fruition of a *self-critique* of

$\frac{3}{H}m_{\#}$, namely, to $\frac{3}{H}mm_{\#} \equiv \Delta(\frac{3}{H}m_{\#})$?

The $s = 3$, $\frac{3}{H}Z_{\#}$ stage/step of this dialectic of Standard Number Systems/Standard Arithmetic Systems, is one dominated by the $\frac{3}{H}m_{\#}$ «*aufheben*» operator, through its operation upon / interaction with, and via its “‘synthesis’” with / “‘hybridization’” with / ‘complex unification’ with / “‘*real subsumption*’” of, every arithmetical axioms-system *component* / kind of number(s) *term* that has already emerged in all previous *steps/stages*. This includes such interaction with the $\frac{3}{H}N_{\#}$ «*arché*»-system, and also with any previous ‘partial uni-thesis’ and/or ‘full uni-thesis’ terms, and with any preceding ‘contra-thesis’ number/arithmetical kinds terms, *including with itself* as previous-stage ‘meta-meristematic’ / “‘vanguard’” ‘contra-thesis’ term -- its own ‘*real self-subsumption*’ / “‘*self-aufheben*» *self-negation*’” -- which yields the new ‘meta-meristematic’ / “‘vanguard’” ‘contra-thesis’ term: $\frac{3}{H}m_{\#} \otimes \frac{3}{H}m_{\#} = \frac{3}{H}m_{\#} \oplus \frac{3}{H}mm_{\#} \equiv \frac{3}{H}m_{\#} \oplus \Delta(\frac{3}{H}m_{\#})$.

This ‘*m*-ization’ of the W -- the addition, to the incipient number “line”, or number “ray”, of the ‘sub-diagram’ [in Background section **B.η** of Part **I**] depicting the W , of the potentially-infinite sequence of “points” representing the “*minus numbers*”, all of them to the left of the “point” representing the *zero(s)*, the “*origin*”, to form the “*Integers*” ‘number-“line”’ of discrete “points” in that figure, and corresponding to the $\frac{3}{H}Z_{\#}$ system -- has worked a further transformation in point-of-view,

beyond the viewpoints native to the $\frac{3}{H}N_{\#}$ and $\frac{3}{H}W_{\#}$ systems before it. It has heightened a sense of ‘*directionality*’ already latent in the “*Natural*” Numbers system, and, even a bit more so, in the “*Whole*” Numbers system, by contrasting the implicit “‘*uni-directionality*’” of the “*Natural*” Numbers, which mount in magnitude to the right, with the apparent presence, for “*Whole*” number “‘analytical geometry’”, on “their” left, of a new “point”, *apparently directionless*, representing the *0s*. Thus, the “*Integers*” system culminates and consolidates this sense of ‘*directionality*’, as a sense of ‘*co-linear bi-directionality*’, in ‘unitation’ with the previously-evoked sense of *non-directionality*, or of ‘*intra-dual directionality*’, of the ± 0 “point”, as the “*origin*” of the number-space. It so culminates and consolidates *within the limits* of a *single* “[pseudo-]line”/dimension of discrete “points”. We might thus suspect that the $s = 3$, stage-emergent ‘self-opposition’ of/to/within the $\frac{3}{H}Z_{\#}$ system might involve an internal breach, or ‘immanent transcendence’, of at least some aspect(s) of these ‘viewpointal’, ‘number-definitional’ deficits & limitations.

The $\frac{3}{H}Z_{\#}$ Integers are, each and all, ‘*Integ[er]al*’, or “*whole*”. The Z number-space is a ‘unitation’ of the ‘*Integ[er]al* negatives’, m , built from the *self-critique* of the *Wholes*’ a , of the ‘*Integ[er]al* neutrals’, built directly from the *Wholes*’ a , and of the ‘*Integ[er]al* positives’ built from the *Wholes*’ aN . The *opposite* kind of number(s) to the Z kind, would therefore be a [sub-]system-category of ‘*non-whole, non-Integ[er]al numbers*’.

We identify/interpret these ‘*non-whole, non-Integ[er]al numbers*’ with/as the [sub-]system of the ‘*Part-ial* numbers’, or of the “‘*fractional*’” numbers, already found extant in our perhaps unsystematic, “chaotic” [Marx] knowledge/experience of the totality of the modern/contemporary ‘Standard Numbers’: the ‘*parts*’ numbers’, or ‘numbers-for-*parts-of-wholes*’/‘numbers-for-*parts-of-integers*’, that we denote herein by $\frac{3}{H}f \mapsto \frac{3}{H}g$. We have thus solved for: $\Delta(\frac{3}{H}m_{\#}) \equiv \frac{3}{H}mm_{\#}$ as

$$\frac{3}{H}f: \vdash \Delta(\frac{3}{H}m_{\#}) = \frac{3}{H}f.$$

Note also that the m are all “negative” in sign. The expected ‘opposite-ness’ of the “‘*fractional*’” *non-Z* numbers to the Z might lead us to expect that the f should therefore all be “positive” in sign. Because the f are built from the m -- because $\frac{3}{H}f \equiv \frac{3}{H}g_{mm}$ -- this turns out, in fact, to be the case, in this one, special

case, in this ‘meta-model’: $f \equiv \{n/d \mid n, d \in m \subset Z\}$, as based upon the standard division operation [here denoted by ‘/’], with both *numerator* & *denominator* “negative” in sign, yields a “positive” *Quotient*. The explicit assertion of zero-exclusion for the d values, in this definition itself, is not needed, because $\pm 0 \notin m$.

Strictly in terms of our “‘analytic-geometric’” vision/visualization of the Standard Arithmetics, in section **B.η.**, **0** filled in a slot to the left of all of the “**Natural**” Numbers, and the fruition of the self-critique of $\frac{3}{H}a_{\#}$ as $\frac{3}{H}m_{\#}$, marked a “filling-in” of a potentially-infinite ‘counter-ray’ of [co-linear] number-space “points”, for number-values that mount in *absolute*-value magnitude in a direction opposite to that in which the **N** mount in magnitude[, & contrary also to the **a**s, which do not mount in magnitude in any direction], with all of the **m** to the left all of the **N**. These first two movements, from **N** to **W**, and then from **W** to **Z**, both involve an “‘extensive filling-in’” -- in the form of a “‘filling-out[wards]’” -- of “‘number-space’”. The emergence of the **f** from out of the self-critique of the **m** marks the onset of a new phase; one also consisting of two consecutive movements -- first, from the **Z** to the **Q**, and, second, from the **Q** to the **R**, as we shall see -- a phase of “‘intensive filling-in’”, in which new ‘number-“points”’ are added-in, in-between every pair of the numbers previously ‘extantized’ explicitly, or ‘outed’, up through **Z**. In particular, the transition from **Z** to **Q**, mediated and effected by $\frac{3}{H}f_{\#}$, packs new ‘number-“points”’ “*densely*”, in-between every pair of ‘number-“points”’ of **Z**. The harbinger of this transformation, $\frac{3}{H}f_{\#}$, in itself merely posits this “*density*” for the **RHS**, ‘positive-signed ray’ of the **Z**-numbers’ “line”-space. But **f**, nevertheless, already portends the comprehensive ‘dense-ification’ and ‘fraction-ization’ of this single “number-line” entire -- negative, neutral, and positive alike -- that arrives in the next stage/step of our ‘meta-model’, stage/step **S = 4**. In any case, the discrete -- vastly porous -- ‘pseudo-line’ of numbers that pertained from **N** to **W** to **Z**, is already, “here” in stage/step **S = 3**, beginning to become an *almost-“real”*, *almost-“solid”*, “*line*” -- a “*dense*”, rectilinear, one-dimensional expanse of ‘number-“points”’, without any apparent lacunae, at least not from the $\frac{3}{H}f_{\#}$ viewpoint.

The Inequation, “‘Well-Formed’” within $\frac{3}{H}Z_{\#}$, that is *Not Solvable within* $\frac{3}{H}Z_{\#}$, and that Therefore Points, from within $\frac{3}{H}Z_{\#}$, to *beyond* $\frac{3}{H}Z_{\#}$ -- to within $\frac{3}{H}Q_{\#}$.

In terms of an immanent critique, or self-critique, of the “*Integers*” system of arithmetic, and, more specifically, in terms of the self-critique of the “minus” numbers [sub-]system of arithmetic, we have noted a family of algebraic inequations, “‘well-formed’” within the algebra of the “*Integers*” arithmetic, which are, however, not “satisfiable” within the “*Integers*” system -- which are not “solvable” by any “*Integer*”. This ‘inequations-family’ can be represented, generically, by the single inequation, $[|x_y \times z| < |z|, |x_y| > \pm 0, z \in \mathbb{Z}]$, via the ‘variable constant’ [via the “parameter”] **Z**, $\forall z \in \mathbb{Z}$. A specific example of this ‘[in]equations-family’ would be the equation $[+2x_y = +1 \mid x_y > \pm 0]$, the solution of which is, of course, $x_y = \pm 1/\pm 2 \notin \mathbb{Z}$. This ‘inequations-family’ represents a paradox from the point of view of a definition of ‘Standard Number’ restricted within the concept of the “*Integers*”: ‘*the paradox of decreasive multiplication*’. The “*Integers*” concept can support the multiplication, by an unknown number, of a known “*Integer*”, that yields that known “*Integer*” unchanged, neither increased or decreased: $x \times z = z, \forall z \in \mathbb{Z}$.

The solution to that equations-family has been available in, and ever since, our «*arché*», our *beginning* system, $\frac{3}{H}N_{\#}$. That solution, in its $\frac{3}{H}Z_{\#}$ -form, is, of course, **+1**, the “multiplicative identity element”: $x = +1$. The “*Integers*” concept can also support ‘*decreasive/increasive/neither multiplication*’ in the sense of ‘annihilatory multiplication’: $x \times z = \pm 0, \forall z \in \mathbb{Z}$. The solution to that equations-family has been available in, and ever since, our *second*, the “*Whole Numbers*”, system, $\frac{3}{H}W_{\#}$.

That solution, in its $\frac{3}{H}Z_{\#}$ -form, is, of course, $\pm 0: x = \pm 0$. It is, of course, ‘*decreasive*’ for $z > \pm 0$, ‘*increasive*’ for $z < \pm 0$, and *neither/“non-increasive-&-non-decreasive”* for $z = \pm 0$. Finally, the “*Integers*” concept can also support a form of ‘*decreasive/increasive/neither multiplication*’, by an unknown number, of a known “*Integer*”, that “‘toggles’”/reverses the sign of that non-zero “*Integer*”: $x \times z = -z$. The solvability of that equations-family first arose, of course, only with our stage/step **S = 3** “*Integers*” system itself, $\frac{3}{H}Z_{\#}$. That solution, in its $\frac{3}{H}Z_{\#}$ -form, is, of course, **-1**: $x = -1$. It is, of course, likewise, ‘*decreasive*’ for $z > \pm 0$, ‘*increasive*’ for $z < \pm 0$. It is also *neither/“non-increasive-&-non-decreasive”* for $z = \pm 0$. But the “*Integers*” cannot supply, for every individual “*Integer*”, **Z**, an ‘*anti-“Integer”*’, another “*Integer*”, which, when multiplied into the non-zero “*Integer*” in question, **Z**, yields a non-zero “*Integer*” which is always smaller, in “absolute value”, than **Z**. Such ‘*decreasive multiplication*’, ‘*subtractive multiplication*’, or ‘*contractive multiplication*’, is beyond the ken of the “*Integers*” «*mentalité*».

Therefore, the self-critique of the “*Integers*”, as seeded by the inequation described above, and, more specifically, the self-critique of ‘the minus numbers’, leads us to the “fractional numbers”, $\frac{3}{H}Q_{\#mm} = \frac{3}{H}f_{\#}$, the *numbers* that fill in “*densely*”, as magnitudes, “*between*” every pair of positive *integers*.

The **f** kind-of-number(s), in this stage’s now “‘formal subsumption’” of/contrast of the **f** kind and/with their [sub-]system of arithmetic, $\frac{3}{H}f_{\#}$, versus the **Z** kind-of-number(s), and their system of arithmetic, $\frac{3}{H}Z_{\#}$, and in terms of our “‘analytic geometric’” visualization of these “number-spaces”, redefine ‘Standard Number’ to be, not just “‘counts’” of *whole, integ[er]al units*, but also “‘counts’” of arbitrary *commensurable parts* of such *units* as well.

This stage of our ‘self-dialogue’ has therefore far further “filled-in” our “analytic-geometric” number-space. It has equipped us conceptually with a new, ‘*part-ial*’, potentially-infinite “‘*back-fill*’” to the preponderance of commensurable *gaps between integers* in our incipient “‘number-line’”; a new “*density*” and ‘*proto-solidity*’ of *numbers* along that “‘number-line’”, complementing and supplementing the original/earlier/surpassed, *nearly-void* number-spaces -- the mostly ‘number-*empty*’ *vacua* -- of the **Z**, the **W**, and the **N** ‘number-*near-voids*’. Thus, the latter are now ‘re-seen’, in this new relative light, as potentially -- & as explicitly, in retrospect -- *vanishingly* ‘sparsely-populated’ rectilinear number arrays, profoundly “‘rarefied’”, discrete, and dis-continuous. Thus, all **3** -- the **Z**, the **W**, and the **N** -- are, together, [re-]visualized, in the light of $\frac{3}{H}f_{\#}$, as **3** successive, progressive primitive predecessor stages to the stage of the “‘*rational numbers*’”; as predecessors consisting of the most minimal, most “‘rarefied’” realizations of number possibility, all of which implicitly inhere in a single, co-linear, rectilinear, but ‘bi-directional’, ‘*massively-more-number-multiplicitous*’ -- ‘*proto-solid*’ -- “*dimension*” of number: in a potentially single, ‘*densely*’-populated’, ‘*proto-continuous*’ “‘number-*axis*’”, namely: **Q**, whose harbinger is **f**.

Taking the Measure of Where We Have Arrived Cognitively via Step/Stage 3 in this Chain of Immanent Critiques.

Thus, we arrive at -- as the fruition of Step/Stage 3 -- the 'mutual controversion' of $\frac{3Z}{H\#}$ vs. $\frac{3f}{H\#}$. We have $\frac{3Z}{H\#}$ as the partial, one-sided, unnecessarily restrictive view of modern 'Standard Number', as limited to "'wholes-counting'", or to "'whole-units-counting'", & to "'non-counting'" of "'part-units'", or of "fractions" of *units*. And we have this $\frac{3Z}{H\#}$, in an 'additively oppositional' relation to $\frac{3f}{H\#}$ as *counter-example* to $\frac{3Z}{H\#}$'s 'number as whole-units-counting, parts-of-units-ignoring' view. That is, we have $\frac{3f}{H\#}$ as a [sub-]system of 'between-positive-integers number-points'. But, in itself, $\frac{3f}{H\#}$ is just as "one-sided"/"partial" -- just as incomplete/inadequate -- as is $\frac{3Z}{H\#}$, if not even more so. The $\frac{3f}{H\#}$ [sub-]system of arithmetic encompasses only "positive fractions", excluding even the 'neutral point' of $Z, \pm 0$, as well as all of the "negative fractions". So $\frac{3Z}{H\#}$ is equally a *counter-example* to $\frac{3f}{H\#}$ -- to any claims of adequacy/completeness/totality that might be 'mis-made' on behalf of $\frac{3f}{H\#}$.

We thus remain in an unsatisfactory situation, in terms of systematically -- including of "'taxonomically'" -- organizing & accounting for our entire experience of the totality of modern, contemporary "Standard Arithmetic(s)", although the 'explicitization' of $\frac{3f}{H\#}$ does mark an advance over the even more unsatisfactory situation wherein we dwelt in Step/Stage 2, and, even more so, wherein we dwelt in the steps/Stages before Step/Stage 2.

But to improve our situation further, we must move on, on to Step/Stage 4, to $\frac{3}{H}\overline{H}^{\#} = \left(\frac{3N}{H\#}\right)^2$, i.e., to the self-«aufheben», i.e., to the self-reflection, or "'squaring with itself'", of Step/Stage 3's own result, namely, of $\left(\frac{3Z}{H\#} \dashv \oplus \frac{3f}{H\#}\right)$ [with] itself, or [shortcut] to the results of the action of '[de]flection' of the collective «aufheben» operator/operation denoted $\left(\frac{3Z}{H\#} \dashv \oplus \frac{3f}{H\#}\right)$ by the «aufheben» operator/operation denoted $\frac{3f}{H\#}$.

What the Symbol ${}^3_H Z_{\#}$ "Intends": *Axioms-System of the Arithmetic of the so-called "Integers", Z* [commenced]

Z0. Z is not the Empty Set. $Z \neq \emptyset \equiv \{ \}$;

First Order Peano Postulates: [mainly] "Phonographic" Rendering	[mainly] "Ideographic" Rendering	[C ΔW];
ZP1. 0 is an "integer".	$0 \in Z$;	
ZP2. The successor of any "integer" is also an "integer".	$z \in Z \rightarrow s(z) \in Z$;	
ZP3. No two, distinct "integers" have the same successor.	$z_1, z_2 \in Z \ \& \ z_1 \neq z_2 \rightarrow s(z_1) \neq s(z_2)$;	
ZP4. 0 is not the successor of any other "integer".	$\neg \exists x \in Z \mid s(x) = 0$; $x = -1$. (0 has a predecessor in Z , and that predecessor is -1 .)	

Second Order Generalized Peano Induction Postulate for "Integers", for Set S as Extension of Intension [Predicate] P

ZP5.a. $[[\forall S \neq \emptyset][\forall z_1, z \in Z][[[z_1 \in S] \ \& \ [[z_1 \in S] \rightarrow [s(z_1) \in S]]] \rightarrow [[\forall z \in Z][[z \geq z_1][z \in S]]]]]$

ZP5.b. $[[\forall S][S \subset Z][[[z \in S] \ \& \ [[z \in S] \rightarrow [[z+1 \in S] \ \& \ [z-1 \in S]]]]] \rightarrow [S=Z]]]$

First Order Axioms of "Integer" Addition:

ZA1. For all z_1, z_2 in Z , $z_1 + z_2 = z_2 + z_1$ [Additive Commutativity];

ZA2. For all z_1, z_2, z_3 in Z , $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ [Additive Associativity];

ZA3. For all z_1, z_2 in Z , $z_1 + z_2$ is in Z [Additive Closure of Z]; $[\forall z_1, z_2 \in Z][z_1 + z_2 \in Z]$;

ZA4. There is an element 0 in Z such that, for all z in Z , $0 + z = z$ [Additive Invariance Element];
 $[\exists 0 \in Z][\forall z \in Z][0 + z = z + 0 = z]$;

ZA5. For every z in Z , there exists $(-z)$ in Z , such that $z + (-z) = 0$ [Additive Inverse Elements] [C ΔW];
 $[\forall z \in Z][\exists -z \in Z][z + (-z) = (-z) + z = 0]$;

First Order Axioms of "Integer" Multiplication:

ZM0. There is an element 0 in Z such that, for all z in Z , $0 \times z = 0$ [Multiplicative Predominance];
 $[\exists 0 \in Z][\forall z \in Z][0 \times z = z \times 0 = 0]$;

ZM1. For all z_1, z_2 in Z , $z_1 \times z_2 = z_2 \times z_1$ [Multiplicative Commutativity];

ZM2. For all z_1, z_2, z_3 in Z , $(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3)$ [Multiplicative Associativity];

ZM3. For all z_1, z_2 in Z , $z_1 \times z_2$ is in Z [Multiplicative Closure of Z]; $[\forall z_1, z_2 \in Z][z_1 \times z_2 \in Z]$;

ZM4. There is an element 1 in Z such that, for all z in Z , $1 \times z = z$ [Multiplicative Invariance Element];
 $[\exists 1 \in Z][\forall z \in Z][1 \times z = z \times 1 = z]$;

What the Symbol ${}^3_H Z_{\#}$ "Intends" [continued & concluded].

First Order Axioms for the "Hybridization" of the "Integer" Operations of "Multiplication" and of "Addition":

ZH1. For all z_1, z_2, z_3 in Z , $z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$;
 $[\forall z_1, z_2, z_3 \in Z][z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3]$ ["Distributivity" ["Hybridization"] of Multiplication over Addition];

ZH2. For all z_1, z_2, z_3 in Z , $(z_1 + z_2) \times z_3 = z_1 \times z_3 + z_2 \times z_3$;
 $[\forall z_1, z_2, z_3 \in Z][(z_1 + z_2) \times z_3 = z_1 \times z_3 + z_2 \times z_3]$ ["Distributivity" ["Hybridization"] of Addition over Multiplication];

First Order Axioms of "Integer" Exponentiation:

ZE0. For all z in $Z - \{0\}$, $z^0 = 1$; the value of the expression 0^0 is undefined [Exponentiation Predominance];

ZE1. For all $z_1, z_2, z_3 > 0$ in Z , $z_1^{z_2} \times z_1^{z_3} = z_1^{z_2 + z_3}$ [Exponentiation Additivity] [C ΔW];

ZE2. For all $z_1, z_2, z_3 > 0$ in Z , $(z_1^{z_2})^{z_3} = z_1^{z_2 \times z_3}$ [Exponentiation Multiplicativity] [C ΔW];

ZE3. For all $z_1, z_2, z_3 > 0$ in Z , $(z_1 \times z_2)^{z_3} = z_1^{z_3} \times z_2^{z_3}$ [Exponentiation "Distributivity"] [C ΔW];

ZE4. For all z_1, z_2 in Z , $z_1^{z_2}$ is in Z [Exponentiation Closure] [C ΔW]; $0^0 \notin Z$; $z_1^{z_2} \notin Z$ if $z_2 < 0$, unless $z_1 = \pm 1$;

ZE5. For all z in Z , $z^1 = z$ [Exponentiation Invariance];

Axioms of "Integer" Order:

ZO1. For all z_1, z_2 in Z , either $z_1 > z_2$, or $z_1 = z_2$, or $z_1 < z_2$ [Trichotomy "Law" [First Order]];
 $[\forall z_1, z_2 \in Z][[z_1 > z_2] \vee [z_1 = z_2] \vee [z_1 < z_2]]$;

ZO2. For all $z_1 > 0, z_2 > 0$ in Z , $z_1 + z_2 > z_1, z_2$ [Additive Order for "Positive" Integers [First Order]] [C ΔW];
 $[\forall z_1, z_2 \in Z][z_1, z_2 > 0][z_1 + z_2 > z_1, z_2]$;

ZO3. For all $z_1 > 0, z_2 > 0$ in Z , $z_1 \times z_2 \geq z_1, z_2$ [Multiplicative Order for "Positive" Integers [First Order]] [C ΔW];
 $[\forall z_1, z_2 \in Z][[[z_1 > 0 < z_2] \rightarrow [z_1 \times z_2 > z_1, z_2]] \ \& \ [[z_1 \times z_2 = 0] \Leftrightarrow [[z_1 = 0] \vee [z_2 = 0]]]$;

ZO4. Any non-empty set of "integers" contains a least element [Well Ordering Principle [Second Order]];
 $[[\forall S \subset Z][[S \neq \emptyset][\exists s \in S][\forall z \in S][z \leq s]]]$;

ZO5. $0 < 1$ [Order of Invariance Elements [First Order]].

C.2.4. stage/step S = 4: Consolidation and ‘self-aporization’ of the System of Arithmetic of the “Rational Numbers”, \mathbf{Q} .

- 4.1. descriptive name of stage: *The Consolidation of the “Rationals” and their “Formal Subsumption” by the “diagonal numbers”*.
- 4.2. stage ‘parametrics’: [total terms count, 2^4] 16; [new terms count, 2^{4-1}] 8; [new terms needing solution count, $(2^{4-1} - 2)$] 6.
- 4.3. «*aporia*» of this stage: $\frac{3}{H} \mathbf{Q}_{\#} \dashv \oplus \frac{3}{H} \mathbf{d}_{\#}$.
- 4.4. “‘incompleteness’”-revealing “unsolvable” “diophantine” equations-family for this stage: $x_8^2 = p, p \in \mathbf{Q}, \& p$ is prime.
- 4.5. “unsolvable” equations’ paradox: *The Paradox of Incommensurability within $\frac{3}{H} \mathbf{f}_{\#}$* .
- 4.6. shortcut rendition of the ‘product-tion’ of stage 4 from stage 3 [via the ‘meta-meristematic principle’]:

$$\begin{aligned} \frac{3}{H} \mathbf{H}_{\#}^4 &= \frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{H}_{\#}^3 = \frac{3}{H} \mathbf{f}_{\#} \otimes \left(\frac{3}{H} \mathbf{H}_{\#}^2 \oplus \frac{3}{H} \mathbf{mN}_{\#} \oplus \frac{3}{H} \mathbf{ma}_{\#} \oplus \frac{3}{H} \mathbf{maN}_{\#} \oplus \frac{3}{H} \mathbf{f}_{\#} \right) = \\ &= \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{H}_{\#}^2 \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{mN}_{\#} \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{ma}_{\#} \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{maN}_{\#} \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{f}_{\#} \right) \oplus \\ &= \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{H}_{\#}^2 \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{mN}_{\#} \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{ma}_{\#} \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{maN}_{\#} \right) \oplus \left(\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{f}_{\#} \right) = \\ &= \frac{3}{H} \mathbf{Q}_{\#} \dashv \oplus \frac{3}{H} \mathbf{d}_{\#}. \end{aligned}$$

- 4.7. ‘pre-solved’ new terms -- new terms of *given* or *already solved* meanings/definitions:

4.7.1. $\left(\frac{3}{H} \mathbf{q}_{\#} \right) \mapsto \mathbb{Q}_{15}$ -- first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’ of the “‘epitome’” of the *Rationals*, $\mathbf{q} \equiv \{ (+0...01/+0...01), (+0...02/+0...01), (+0...03/+0...01), \dots \}$, the sub-space of the \mathbf{Q}

number-space which expresses the fruition of the *critique* of $\frac{3}{H} \mathbf{maN}_{\#}$ by $\frac{3}{H} \mathbf{f}_{\#}$ in the product $\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{maN}_{\#} \equiv \frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{z}_{\#}$, namely $\frac{3}{H} \mathbf{fmaN}_{\#} \equiv \frac{3}{H} \mathbf{q}_{\#}$, the sub-space of the \mathbf{Q} number-space containing all $\frac{3}{H} \mathbf{f}_{\#}$ -converted $\frac{3}{H} \mathbf{m}_{\#}$ -converted $\frac{3}{H} \mathbf{a}_{\#}$ -converted “*Natural*” Numbers, $\mathbf{N} \equiv \{ \mathbf{I}, \mathbf{II}, \mathbf{III}, \dots \}$.

4.7.2. $\left(\frac{3}{H} \mathbf{d}_{\#} \right) \mapsto \mathbb{Q}_{16}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’ of the “‘diagonal’” Numbers, $\mathbf{d} \equiv \{ \text{“irrationals” of } \mathbf{R}_+, \text{ expressed via potentially-infinite/never-repeating decimals} \} \subset \mathbf{R}_+ \subset \mathbf{R}$; fruition of the [immanent] *self-critique* of $\frac{3}{H} \mathbf{f}_{\#}$, namely $\frac{3}{H} \mathbf{ff}_{\#} \equiv \frac{3}{H} \mathbf{d}_{\#}$, so that \mathbf{d} connotes the *positive irrationals*, since \mathbf{f} connotes the *positive rationals*, \mathbf{Q}_+ .

- 4.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module ζ_1]:

4.8.1. $\left(\frac{3}{H} \mathbf{fN}_{\#} \right) \mapsto \mathbb{Q}_9$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘*fraction-ized*’ “*Natural*” Numbers, $\mathbf{fN} \equiv \{ (\mathbf{II}/\mathbf{I}), (\mathbf{II}/\mathbf{I}), (\mathbf{III}/\mathbf{I}), \dots \}$, the sub-space of the \mathbf{Q} number-space which expresses the fruition of the *critique* of $\frac{3}{H} \mathbf{N}_{\#}$ by $\frac{3}{H} \mathbf{f}_{\#}$ in the product $\frac{3}{H} \mathbf{f}_{\#} \otimes \frac{3}{H} \mathbf{N}_{\#}$, namely $\frac{3}{H} \mathbf{fN}_{\#}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\frac{3}{H} \mathbf{N}_{\#}$ with/with/by/by/to/and $\frac{3}{H} \mathbf{f}_{\#}$ as its elevation into \mathbf{Q} .

4.8.2. $\equiv (\text{H}^{\#}_{\text{fa}}) \mapsto \boxed{10}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘*fraction-ized*’ ‘aught’ Numbers, $\mathbf{fa} \equiv \{(\mathbf{I} - \mathbf{I})/1, (\mathbf{II} - \mathbf{II})/1, (\mathbf{III} - \mathbf{III})/1, \dots\}$, the sub-space of the \mathbf{Q} number-space which expresses fruition for the *critique* of $\mathbf{H}_{\#}^{\mathbf{a}}$ by $\mathbf{H}_{\#}^{\mathbf{f}}$ in the product $\mathbf{H}_{\#}^{\mathbf{f}} \otimes \mathbf{H}_{\#}^{\mathbf{a}}$, namely $\mathbf{H}_{\#}^{\mathbf{fa}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment...’” of $\mathbf{H}_{\#}^{\mathbf{a}}$ with/by/to... $\mathbf{H}_{\#}^{\mathbf{f}}$ as $\mathbf{H}_{\#}^{\mathbf{a}}$ ’s elevation into \mathbf{Q} .

4.8.3. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{faN}} \right) \right) \Rightarrow \left(\text{J}_{11} \right)$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

'sub-arithmetic' of the '*fraction-ized*' '*aught-ized* **N**atural Numbers', **faN** $\equiv \{ (0...01/0...01), (0...02/0...01), (0...03/0...01), \dots \}$,

the sub-space of the **Q** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{aN}}$ by $\overset{3}{\text{H}}\overset{\text{f}}{\text{H}}_{\#}$ in the product $\overset{3}{\text{H}}\overset{\text{f}}{\text{H}}_{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{aN}}$,
namely $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{f aN}}$, which is also the “hybridization/synthesis/subsumption/conversion/adjustment/reconciliation” of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{aN}}$
with/with/by/by/to/and $\overset{3}{\text{H}}\overset{\text{f}}{\text{H}}_{\#}$ as $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{aN}}$ ’s elevation into **Q**.

4.8.4. $\equiv \langle H^{\#}_{\text{fm}} \rangle \vdash_{12}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

“‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\overset{3}{\mathbf{H}}\overset{\mathbf{m}}{\#}$ with/by/to/and $\overset{3}{\mathbf{H}}\overset{\mathbf{f}}{\#}$ as its elevation into \mathbf{Q} .

4.8.5. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{N}} \right) \right) \mapsto \left(\left(\text{fm} \right) \right)_{13}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘*fraction-ized*’ ‘*sign-ized*’ **N**atural Numbers’, $\mathbf{fmN} \equiv \{ (+1/+1), (+2/+1), (+3/+1), \dots \}$, the sub-space of the **Q** number-space which expresses the fruition of the critique of $\mathbf{H}^{\#}_{\mathbf{mN}}$ by $\mathbf{H}^{\#}_{\mathbf{f}}$ in the product $\mathbf{H}^{\#}_{\mathbf{f}} \otimes \mathbf{H}^{\#}_{\mathbf{mN}}$, namely $\mathbf{H}^{\#}_{\mathbf{fmN}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\mathbf{H}^{\#}_{\mathbf{fmN}}$ with/with/by/by/to/and $\mathbf{H}^{\#}_{\mathbf{f}}$ as the elevation of $\mathbf{H}^{\#}_{\mathbf{mN}}$ into **Q**.

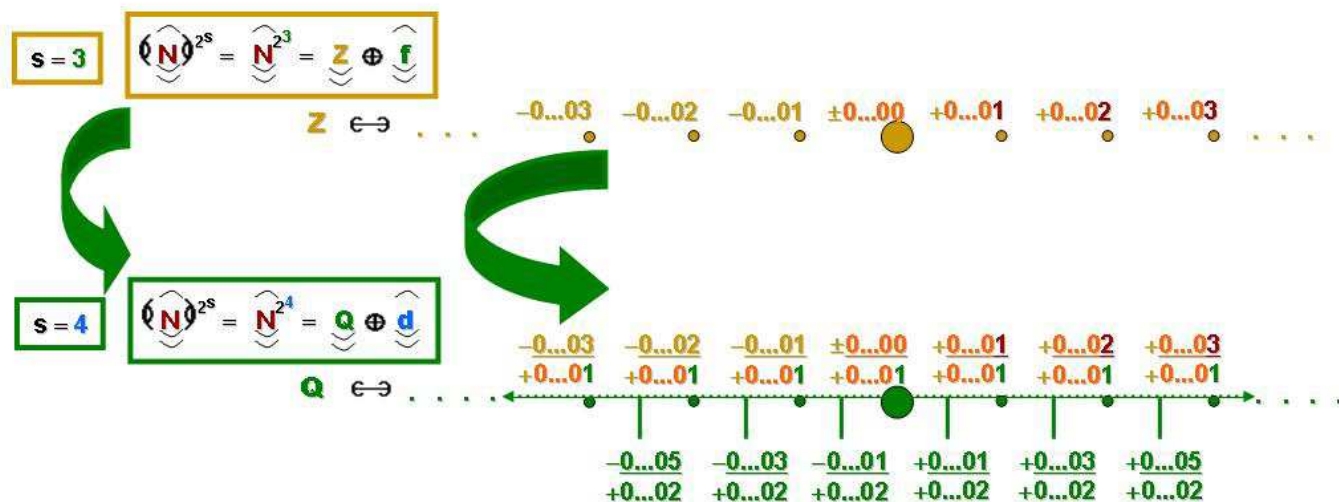
4.8.6. $\equiv \left(\overset{3}{\underset{\text{H}}{\text{O}}} \overset{\#}{\text{fma}} \right) \mapsto \boxed{\text{U}}_{14}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the '*fraction-ized*' '*sign-ized* **a**ught Numbers', **fma** $\equiv \{ \pm(\mathbf{I} - \mathbf{I})/(\mathbf{+1}) = (\pm\mathbf{0}/\mathbf{+1}), \pm(\mathbf{II} - \mathbf{II})/(\mathbf{+1}) = (\pm\mathbf{0}/\mathbf{+1}), \dots \}$,

the sub-space of **Q** number-space which expresses the fruition of the critique of $\overset{\#}{\underset{\text{ma}}{\underset{\text{H}}{\text{H}}}\text{Q}}$ by $\overset{\#}{\underset{\#}{\text{H}}}\text{f}$ in the product $\overset{\#}{\underset{\#}{\text{H}}}\text{f} \otimes \overset{\#}{\underset{\text{ma}}{\underset{\text{H}}{\text{H}}}\text{Q}}$, namely $\overset{\#}{\underset{\text{ma}}{\underset{\text{H}}{\text{H}}}\text{Qf}}$, which is also the “hybridization/synthesis/subsumption/conversion/adjustment/reconciliation” of $\overset{\#}{\underset{\text{ma}}{\underset{\text{H}}{\text{H}}}\text{Q}}$ with/by/to/and $\overset{\#}{\underset{\#}{\text{H}}}\text{f}$ as the elevation of $\overset{\#}{\underset{\text{ma}}{\underset{\text{H}}{\text{H}}}\text{Q}}$ into **Q**.

Number-Line Cumulative Progression Analytic-“Geometric” Model
of the **Gödelian Dialectic of the Standard Arithmetics**:

The Transition from $s = 3$ to $s = 4$, and from $\mathbb{Z} \oplus \dots$ to $\mathbb{Q} \oplus \dots$



Narrative Commentary for stage S = 4. ¿A Supplementary Opposite to the “*Rationals*”, *Q*? The stage 4 ‘ideo-cumulum’ of our ‘meta-model’ ‘super[im]poses’ 16 terms upon one another, the first 8 of which reproduce stage 3, and the last 8 of which are the terms incremental to stage 4. The summed first 7 of those 8 incremental terms --

$$\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fN} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fa} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{faN} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fm} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fmN} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fma} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fmaN} \end{matrix}$$

-- added to the stage 3 sum, representing the “*Integers*” system of arithmetic, $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Z \end{matrix}$, and its counter-example, $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix}$, represent, we find, the ‘*Rational Arithmetic*’ --

$$\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix} = \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Z \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fN} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fa} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{faN} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fm} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fmN} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fma} \end{matrix} \oplus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ \text{fmaN} \end{matrix}$$

We so find, because the first 7 terms connote the “*Integers*” system of arithmetic, $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Z \end{matrix}$, which is “contained in” the “*Rationals*” system of arithmetic, $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix}$, and such that $Z = \{ \dots, -3, -2, -1, \pm 0, +1, +2, +3, \dots \}$, whereas $Q = \{ \dots -3/+1 \dots -2/+1 \dots -1/+2 \dots \pm 0/+1 \dots +1/+2 \dots +2/+1 \dots +3/+1 \dots \}$, and the last 8 of the 9 RHS terms summed in the equation just above represent the ‘fractional numbers’ sub-system, $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix}$, plus its ‘fractional-conversion’/‘[fraction]-ization’ of all previous system-of-arithmetic components, so that the entire “‘space’” of arithmetic axioms-system components is thereby elevated into the “‘fractional’” viewpoint.

The first 15 terms, taken together as connoting $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix}$, form a new «*aporia*», $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix} \oplus \Delta \left(\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \right)$, with the 16th and final term for stage 4: $\Delta \left(\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \right) = \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ ff \end{matrix}$.

¿What, then, *should* this 16th term mean? ¿Is there a kind of arithmetic, that we have encountered in our, perhaps “chaotic” [cf. Marx], ‘untheorized’ experience of the totality of the modern/contemporary “Standard Arithmetics”, based upon a kind of number(s), and an arithmetical ‘ideo-phenomenon’ supported by that kind-of-number(s), which is *opposite*, in kind, in quality, to the ‘ideo-phenomena’ of the “*Rationals*” system *as a whole* -- not just to those of some specific, individual “*ratio*[-*nal*]” number, but to the whole “*genre*” of number “*Ratio-nality*”? ¿More specifically, is there a kind-of-number(s) that is *opposite* to the $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix}$ kind-of-number(s),

that corresponds to the fruition of a *self-critique* of $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix}$, namely, to $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ ff \end{matrix} = \Delta \left(\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \right)$?

The $s = 4$, $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix}$ stage/step of this dialectic of Standard Number Systems/Standard Arithmetic Systems, is one dominated by the $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix}$ «*aufheben*» operator, through its operation upon / interaction with, and via its “‘synthesis’” with / “‘hybridization’” with / ‘complex unification’ with / “‘*real subsumption*’” of, every arithmetical axioms-system *component*/kind-of-number(s) *term* that was already evoked in all previous steps/stages, including of $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix}$ itself. This includes such interaction with the $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ N \end{matrix}$ «*arché*»-system, and also with all previous ‘partial uni-thesis’ and ‘full uni-thesis’ terms, and with all preceding ‘contra-thesis’ number(s)/arithmetic kinds terms, *including with itself* as previous-stage ‘meta-meristemal’/“‘vanguard’” ‘contra-thesis’ term -- its own ‘*real self-subsumption*’/‘*self*’-«*aufheben*» *self-negation*’ -- which yields the new ‘meta-meristemal’/“‘vanguard’” ‘contra-thesis’ term: $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \otimes \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} = \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \ominus \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} = \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ ff \end{matrix} = \begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \ominus \Delta \left(\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ f \end{matrix} \right)$.

This ‘*f*-ization’ of the Z -- the addition, to the incipient number “line” of the $s = 3$ ‘sub-diagram’ [in Background section B.η. of Part I.] depicting the Z , of the potentially-infinite sequence of “points” representing the “fractional numbers”, between every pair of “points” representing “*Integers*”, to form the $s = 4$ sub-diagram of that diagram, for the “*Rationals*” ‘number-“line”’, “*dense*” and *seemingly* “‘solid’”, *replete* with ‘number-points’, and corresponding to the $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix}$ system -- has

worked a further transformation in point-of-view, beyond the viewpoints native to the $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ N \end{matrix}$ and $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ W \end{matrix}$ and $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Z \end{matrix}$ systems before it. It has asserted a view that, not only *whole*, ‘*integ[er]al*’ *units* “count”, and “can count”, but also that [commensurable] *parts* of such *units* -- fractional, ‘*ratio*-nal’ *parts* -- “count”, and “can count”. It has heightened a sense of ‘*number ubiquity*’, and of ‘*number density*’, through the comprehensive, almost-unrestricted application of the arithmetical division operation among pairs of “*Integers*”, with the sole restriction of the *unsolvability* -- the ‘*un*semanticity’ -- of division by the integer ± 0 . Thus, this “*Rational*” Numbers system begins this explicit recognition of the ‘*part*-iality’ of “‘counting number’”, within which the ‘*integ[er]*-ality’ of “‘counting number’” is but the rarest of exceptions, in terms of the proportion of the sub-totality of the $+1$ -[“lowest-common-”]denominated-fractions to the totality of ‘*ratio*-nal’ numbers of which the former forms such a tiny *part*. It so initiates *within the limits* of a single “line”/dimension, now seen as *densely*-packed with *division*-derived ‘number-points’. We might thus suspect that the $s = 4$ stage’s emergent ‘self-opposition’ of/to/within the $\begin{matrix} 3 \\ H \end{matrix} \begin{matrix} \# \\ Q \end{matrix}$ system might involve an internal breach, or ‘immanent transcendence’, of these very ‘viewpointal’, ‘number-definitional’ deficits and limitations.

The $\frac{3}{H_{\#}}Q$ “*Rationals*” are, each and all, ‘*Ratio-nal*’, or “*fraction-al*”, expressible as the *division* of two “*integers*”, one a “numerator”, the other a “denominator”, with the latter “divided into” that the former [excluding only ± 0 as a denominator, but not as a numerator, and including $+1$ as the denominator for ‘*integ[e]ral, whole number rationals*’]. The Q number-space is a ‘unitation’ of the ‘*Integers*’, Z , of the ‘*fractionals*’, f , and of the ‘fractionalizations’ of every “inherited” component of the Z space. The arithmetic system $\frac{3}{H_{\#}}Q$ is $\frac{3}{H_{\#}}f \otimes \frac{3}{H_{\#}}Z$, the ‘ $\frac{3}{H_{\#}}f$ -ization’, or «*aufheben*» ‘ $\frac{3}{H_{\#}}f$ -elevation’, of $\frac{3}{H_{\#}}Z$, in all of its components, component-by-component, and which already includes the “‘evolute’”, [double-]«*aufheben*» *conservation* of $\frac{3}{H_{\#}}Z$.

The *opposite* kind-of-number(s) to the Q kind, would therefore evidently be a [sub-]system-category of ‘*non-Ratio-nal, non-fraction-al, non-division-al numbers*’; of “*incommensurable*” *parts* of *unit[y](s)*, constituting a “‘determinate negation’” category of the Q category, by co-negating the Q category’s ‘*integer-Ratio-nal*’, ‘*integer-fraction-al*’, ‘*integer-division-al*’, and “*commensurable*” “determinations”.

We identify/interpret these ‘*non-Ratio-nal, non-fraction-al numbers*, with/as the [sub-]system of the ‘*diagonal numbers*’, or of the “potentially-infinite/never-repeating *decimal numbers*”, i.e., with/as the ‘*positive ir-ratio-nal numbers*’, *algebraic & transcendental* alike, that we have already found extant in our perhaps *un*systematic, “chaotic” [Marx] knowledge/experience of the totality of the modern/contemporary ‘Standard Numbers’: the kind of number(s) that we denote herein by:

$$\frac{3}{H_{\#}}d \mapsto \frac{3}{H_{\#}}d_{16}$$

We have thus solved for the meaning of the ‘*contra-thesis*’ category of the “‘positive *fractional numbers*’” ‘*contra-thesis*’ category *itself* --

$$\Delta\left(\frac{3}{H_{\#}}f\right) = \frac{3}{H_{\#}}\frac{3}{H_{\#}}f \text{ as } \frac{3}{H_{\#}}d; \vdash \Delta\left(\frac{3}{H_{\#}}f\right) = \frac{3}{H_{\#}}d.$$

We call these ‘*positive ir-rational numbers*’ the ‘*diagonal numbers*’, **(1)** because it was the recognition of the incommensurability of the length of the *d* diagonal of a square with the length of that square’s side, among the ancient Pythagoreans, that represents humanity’s first documented encounter with number ‘*ir-ratio-nality*’ in the Ancient Occident that is known to us, & **(2)** due to Cantor’s “*diagonal argument*”, regarding the vastly greater ubiquity of the ‘*ir-ratio-nals*’ vis-à-vis the ‘*ratio-nals*’.

Recall also that the f are all “positive” in sign. Because the d are built from the f -- because $\frac{3}{H_{\#}}d = \frac{3}{H_{\#}}\frac{3}{H_{\#}}f$ -- the d also are strictly positive in this ‘meta-model’.

Strictly in terms of our “‘analytic-geometric’” vision/visualization of the Standard Arithmetics, addressed in section **B.η.**, the emergence of the a from out of the self-critique of the N filled in a slot to the left of all of the “Natural” Numbers “points”, and the fruition of the *self-critique* of $\frac{3}{H_{\#}}a$ as $\frac{3}{H_{\#}}m$, marked a “filling-in” of a potentially-infinite ‘counter-ray’ of ‘number-“points”’, again all to the left of all of the “Natural” Numbers “points”. These first two movements, from the N to the W , and then from the W to the Z , both involve an “‘extensive filling-out’” -- in the form of a “‘filling-out[word]’” -- of number-space. The emergence of the f from out of the self-critique of the m marked the onset of a new phase; one also consisting of two consecutive movements -- first, from the Z to the Q , and, second, from the Q to the R , the latter having the d as their *limen* -- a phase of “‘intensive filling-in’”, in which new ‘number-“points”’ are added-in, in-between every pair of the ‘number-“points”’ previously ‘extantized’ explicitly, or ‘outed’, for the Z , and, then, next, even more so, for the Q . In particular, the transition from the Q to the R , mediated and effected by $\frac{3}{H_{\#}}d$, packs new ‘number-“points”’ “*continuously*”, in-between each pair of those of the “points” of Q -- the ‘ Q whole-number-“points”’ -- which have nothing other than a potentially-infinite succession of 0 s to the right of their *d* decimal points in their *d* decimal representations. The harbinger of this transformation, $\frac{3}{H_{\#}}d$, in itself merely posits this “*continuity*” for the **RHS**, ‘positive-signed ray’ of the Q -numbers’ “line”-space.

But the advent of the d , nevertheless, already portends the comprehensive ‘*continuity*’ and ‘*potentially-infinite-d decimal-ization*’ of this single “number-line” entire, negative, neutral, and positive alike, that arrives in the next *stage/step* of our ‘meta-model’, *stage/step* **S = 5**. In any case, the only at last ‘*numbers-dense*’ -- but all along and still vastly ‘por[e]-ous’ -- ‘*pseudo-solid*’ “line” of numbers that pertained from the N to the W to the Z and, at last, even to the Q , is, already, “here”, in *stage/step* **S = 4**, even with its merely “‘*formal subsumption*’” of $\frac{3}{H_{\#}}Q$ by $\frac{3}{H_{\#}}d$, on the *liminal verge* of becoming a “*real*”, “*solid*”, true/“full” “*line*” -- a

“*continuous*”, rectilinear, **1-D** expanse “of [number-]points”, *without any more lacunae*, at least not from the $\frac{3}{H_{\#}}d$ viewpoint -- or from the viewpoint of the “Standard” $\frac{3}{H_{\#}}R$.

The Equation, “‘Well-Formed’” within $\frac{3}{H}Q$, that is *Not Solvable* within $\frac{3}{H}Q$, and that Therefore Points, from within $\frac{3}{H}Q$, to beyond $\frac{3}{H}Q$ -- to within $\frac{3}{H}R$.

In terms of an immanent critique, or self-critique, of the “*Rationals*” system of arithmetic, and, more specifically, in terms of the self-critique of the “fractional” numbers [sub-]system of arithmetic, we have noted a family of algebraic equations, “‘well-formed’” within the algebra of the “*Rationals*” arithmetic, which are, however, *not* “satisfiable” within the “*Rationals*” system -- which are *not* “solvable” by any “*Rational*” number. This ‘equations-family’ can be represented, generically, by the single equation, $[x_8^2 = p, 0 < p \in Q]$, with p denoting a [positive] [rational] prime number. A specific instance of this equations-family is $x_8^2 = +2/+1$, i.e., $x_8 = \pm\sqrt{+2}$. This equation, not by immediate inspection, but upon thorough analysis, is found to assert a paradox for a definition of ‘Standard Number’ restricted to the concept of the “*Rationals*”, ‘the paradox of incommensurability’. The “*Rationals*” concept of number can support, e.g., the square-root “root-extraction” operation on *some* of its positive composite rational numbers, and even on *some* of its positive fractional numbers, but the square-root root-extraction operation applied to [positive] “prime” “rational numbers” never yields a “rational number”. Such square-root “root-extraction” operations are beyond the ken of the “*Rationals*” «*mentalité*».

Therefore, the self-critique of the “*Rationals*”, as seeded by the equation described above, and, more specifically, the self-critique of ‘the fractional numbers’, leads us to the “diagonal numbers”, $\frac{3}{H}Q_{ff} \equiv \frac{3}{H}d$, the numbers that fill in “continuously”, as magnitudes, “‘among’” the positive rationals.

The d kind-of-numbers, in this stage’s now “‘formal subsumption’” of/contrast with the d kind & their [sub-]system of arithmetic, $\frac{3}{H}d$, vs. the Q kind-of-number(s), and their system of arithmetic, $\frac{3}{H}Q$, and in terms of our “‘analytic geometric’” visualization of these “number-spaces”, redefine ‘Standard Number’ to be, not just “‘counts’” via “commensurable” parts of units, but also “‘counts’” -- or at least ‘a-mounts’ -- via “incommensurable” parts of units as well.

This stage of our ‘self-dialogue’ has therefore far *far* further “filled-in” our “analytic-geometric” number-space. It has equipped us conceptually with a *new-part-ial*, potentially-infinite within already potentially-infinite “‘back-fill’” to the preponderance of “incommensurable” gaps among rationals in our incipient “‘line[-up]’” of numbers; a new “continuity” and ‘solidity’ of numbers along that “‘number-line’”, complementing and supplementing the original/earlier/surpassed, *gaping with gaps* number-spaces -- the *number-‘sparsenesses’* -- of *even the Qs*, and, of course, also of the *Zs*, the *Ws*, and the *Ns* ‘number-near-voids’. Via the advent of $\frac{3}{H}d$, the latter are now ‘re-seen’, in *this new relative light*, as potentially -- and as explicitly, in retrospect -- *vanishingly*-‘sparsely-populated’ rectilinear number arrays, profoundly “‘rarefied’”, discrete, and dis-continuous. Thus, all four are, together, [re-]visualized, in the light of $\frac{3}{H}d$, as four successive stages of the most minimal, rarest realizations of number possibility, all of which inhere in a single, co-linear, rectilinear, *massively-more-number-‘multiplicitous’* -- “‘solid’” -- “dimension” of number: potentially a single, ‘continuously-populated’ “‘number-axis’”, namely: R , whose harbinger is d .

Taking the Measure of Where We Have Arrived Cognitively via Step/Stage 4 in this Chain of Immanent Critiques.

Thus, we arrive at -- as the fruition of Step/Stage 4 -- the ‘mutual controversion’ of $\frac{3}{H}Q$ vs. $\frac{3}{H}d$. We have $\frac{3}{H}Q$ as the partial, one-sided, unnecessarily restrictive view of modern ‘Standard Number’ as limited to “‘fractional-counting’”, or to “‘commensurable-parts-of-units-counting’”, and to the “‘non-counting’” of “‘ir-ratio-nal parts’” of units. And we have this $\frac{3}{H}Q$, in an ‘additively oppositional’ relation to $\frac{3}{H}d$ as *counter-example* to $\frac{3}{H}Q$ ’s ‘number as *commensurable-parts-of-units-counting*, *ir-ratio-nal parts-of-units-ignoring*’ view. That is, we have $\frac{3}{H}d$ as a [sub-]system of ‘among-positive-rational number-“points”’.

However, in itself, $\frac{3}{H}d$ is just as “one-sided”/“partial” -- just as incomplete/inadequate -- as $\frac{3}{H}Q$, if not even more so. The $\frac{3}{H}d$ [sub-]system of arithmetic encompasses *only* “positive *ir-ratio-nals*”, excluding even the ‘neutral point’ of Q , $\pm 0/+1$, as well as all of the “negative *ir-ratio-nals*”. So $\frac{3}{H}Q$ is equally a *counter-example* to $\frac{3}{H}d$ -- to any claims of adequacy/completeness/totality that might be ‘mis-made’ on behalf of $\frac{3}{H}d$. We thus remain in an unsatisfactory situation, in terms of systematically -- including of “‘taxonomically’” -- organizing & accounting for our entire experience of the totality of modern, contemporary “Standard Arithmetic(s)”, although the ‘explicitization’ of $\frac{3}{H}d$ does mark an advance over the even more *unsatisfactory* situation wherein we dwelt in Step/Stage 3, and, even more so, wherein we dwelt in the Steps/Stages before Step/Stage 3.

But to improve our situation further, we must move on, on to Step/Stage 5, to $\frac{3}{H}H_5 = (\frac{3}{H}N_5)^{2^5}$, to the *self-«aufheben»*, i.e., to the *self-reflection*, or “‘squaring with itself’”, of Step/Stage 4’s own result, namely, of $(\frac{3}{H}Q \dashv \oplus \frac{3}{H}d)$ [with] itself, or [shortcut] to the results of the action of ‘[de]flection’ of the collective «*aufheben*» operator/operation denoted $(\frac{3}{H}Q \dashv \oplus \frac{3}{H}d)$ by the «*aufheben*» operator/operation denoted $\frac{3}{H}d$.

What the Symbol $\overset{3}{\underset{H}{\mathbb{Q}}}_{\#}$ "Intends": *Axioms-System of the Arithmetic of the so-called "Rational" Numbers, \mathbb{Q}* [commenced]

Q0. \mathbb{Q} is not the Empty Set. $\mathbb{Q} \neq \emptyset \equiv \{ \}$;

<p>First Order Peano Postulates (mainly) "Phonogramic" Rendering:</p> <p>QP1. 0 is an "Integer".</p> <p>QP2. The <i>successor</i> of any "Integer" is also an "Integer".</p> <p>QP3. No two, distinct "Integers" have the same <i>successor</i>.</p> <p>QP4. $\neg [0 \text{ is the } \textit{successor} \text{ of any other "Integer"}]$.</p> <p>Second Order Generalized Peano Inductive Postulate for "Rationals", for Set S as Extension of Intension (Predicate) P:</p> <p>QP5 $[[\forall s \neq 0][\forall q_1, q_2 \in \mathbb{Q}][[(q_1 \in S) \& [(q_2 \in S) \Rightarrow [s(q_1) \in S]]] \Rightarrow [(q_2 \in \mathbb{Q}) \Rightarrow [q_2 \geq q_1][q_2 \in S]]]]]$</p>	<p>(mainly) "Ideogramic" Rendering:</p> <p>$0 \in \mathbb{Q}$.</p> <p>$q \in \mathbb{Q} \Rightarrow s(q) \in \mathbb{Q}$.</p> <p>$q_1, q_2 \in \mathbb{Q} \wedge q_1 \neq q_2 \Rightarrow s(q_1) \neq s(q_2)$.</p> <p>$\nexists x \in \mathbb{Q} \mid s(x) = 0$. $x = -1$ (0 has a <i>predecessor</i> in \mathbb{Q}, and that <i>predecessor</i> is -1).</p>	<p>[$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]:</p> <p>[$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]:</p>
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First Order Axioms of "Rational" Addition:

- QA1.** For all q_1, q_2 in \mathbb{Q} , $q_1 + q_2 = q_2 + q_1$ [Additive Commutativity].
- QA2.** For all q_1, q_2, q_3 in \mathbb{Q} , $(q_1 + q_2) + q_3 = q_1 + (q_2 + q_3)$ [Additive Associativity].
- QA3.** For all q_1, q_2 in \mathbb{Q} , $q_1 + q_2$ is in \mathbb{Q} [Additive Closure of \mathbb{Q}]; $[\forall q_1, q_2 \in \mathbb{Q}][q_1 + q_2 \in \mathbb{Q}]$.
- QA4.** There is an element 0 in \mathbb{Q} such that, for all q in \mathbb{Q} , $0 + q = q$ [Additive Invariance Element].
- QA5.** For every q in \mathbb{Q} , there exists $(-q)$ in \mathbb{Q} , such that $q + (-q) = 0$ [Additive Inverse Elements].
- $[\forall q \in \mathbb{Q}][\exists -q \in \mathbb{Q}][q + (-q) = (-q) + q = 0]$.

First Order Axioms of "Rational" Multiplication:

- QM0.** There is an element 0 in \mathbb{Q} such that, for all q in \mathbb{Q} , $0 \times q = 0$ [Multiplicative Predominance].
- $[\exists 0 \in \mathbb{Q}][\forall q \in \mathbb{Q}][0 \times q = q \times 0 = 0]$.
- QM1.** For all q_1, q_2 in \mathbb{Q} , $q_1 \times q_2 = q_2 \times q_1$ [Multiplicative Commutativity].
- QM2.** For all q_1, q_2, q_3 in \mathbb{Q} , $(q_1 \times q_2) \times q_3 = q_1 \times (q_2 \times q_3)$ [Multiplicative Associativity].
- QM3.** For all q_1, q_2 in \mathbb{Q} , $q_1 \times q_2$ is in \mathbb{Q} [Multiplicative Closure of \mathbb{Q}]; $[\forall q_1, q_2 \in \mathbb{Q}][q_1 \times q_2 \in \mathbb{Q}]$.
- QM4.** There is an element 1 in \mathbb{Q} such that, for all q in \mathbb{Q} , $1 \times q = q$ [Multiplicative Invariance Element].
- QM5.** For every $q \neq 0$ in \mathbb{Q} , there is $(1/q)$ in \mathbb{Q} , such that $q \times (1/q) = 1$ [Multiplicative Inverse Elements] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$];
- $[\forall q \in \mathbb{Q}][\exists 1/q \in \mathbb{Q}][q \times (1/q) = (1/q) \times q = 1]$; the value of the expression $(q/0)$ is undefined;

What the Symbol $\overset{3}{\underset{H}{\mathbb{Q}}}_{\#}$ "Intends" [continued & concluded].

First Order Axioms for the "Hybridization" of the "Rational" Operations of "Multiplication" and of "Addition":

- QH1.** For all q_1, q_2, q_3 in \mathbb{Q} , $q_1 \times (q_2 + q_3) = q_1 \times q_2 + q_1 \times q_3$ ["Distributivity" ["Hybridization"] of Multiplication over Addition].
- QH2.** For all q_1, q_2, q_3 in \mathbb{Q} , $(q_1 + q_2) \times q_3 = q_1 \times q_3 + q_2 \times q_3$ ["Distributivity" ["Hybridization"] of Addition over Multiplication].

Axioms of "Rational" Exponentiation:

- QE0.** For all q in $\mathbb{Q} - \{0\}$, $q^0 = 1$; the value of the expression 0^0 is undefined [Exponentiation Predominance].
- QE1.** For all $q_1, q_2, q_3 \geq 1$ in \mathbb{Q} , $q_1^{q_2 \times q_3} = q_1^{q_2 \times q_3}$ [Exponentiation Additivity] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]; some values $\notin \mathbb{Q}$.
- QE2.** For all $q_1, q_2, q_3 \geq 1$ in \mathbb{Q} , $(q_1^{q_2})^{q_3} = q_1^{q_2 \times q_3}$ [Exponentiation "Multiplicativity"] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]; some values $\notin \mathbb{Q}$.
- QE3.** For all $q_1, q_2, q_3 \geq 1$ in \mathbb{Q} , $(q_1 \times q_2)^{q_3} = q_1^{q_3} \times q_2^{q_3}$ [Exponentiation "Distributivity"] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]; some values $\notin \mathbb{Q}$.
- ~~**QE4.** For all q_1, q_2 in \mathbb{Q} , $q_1^{-q_2}$ is in \mathbb{Q} [Exponentiation Closure] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]; $0^0 \notin \mathbb{Q}$; $\exists q_1 \in \mathbb{Q} \mid q_1^{q_2} \notin \mathbb{Q}$ if $q_2 \in (0, 1)$, etc.~~
- QE5.** For all q in \mathbb{Q} , $q^1 = q$ [Exponentiation Invariance].
- QE6.** For all q_1, q_2, q_3 in \mathbb{Q} , $(q_2/q_1)^{-q_3} = (q_1/q_2)^{q_3} = q_1^{q_3}/q_2^{q_3}$; $(q_3^{q_2}/q_3^{q_1}) = q_3^{q_2 - q_1}$ [Exponentiation Rationality] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$]; some $\notin \mathbb{Q}$.

Axioms of "Rational" Order:

- QO1.** For all q_1, q_2 in \mathbb{Q} , either $q_1 > q_2$, or $q_1 = q_2$, or $q_1 < q_2$ [Trichotomy "Law" [First Order]].
- QO2.** For all $q_1 > 0, q_2 > 0$ in \mathbb{Q} , $q_1 + q_2 > q_1, q_2$ [Additive Order for "Positive" Rationals [First Order]].
- QO3.** For all $q_1 > 1, q_2 > 1$ in \mathbb{Q} , $q_1 \times q_2 > q_1, q_2$ [Multiplicative Order, Trans-Unit' Rationals [First Order]] [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$];
- $[\forall q_1, q_2 \in \mathbb{Q}][[(q_1 > 1 \& q_2 > 1) \Rightarrow [q_1 \times q_2 > q_1, q_2]] \& [[q_1 \times q_2 = 0] \Leftrightarrow [(q_1 = 0) \vee (q_2 = 0)]]]$
- QO4.** Any non-empty set of "Rationals" contains a *least* element [Well Ordering Principle [Second Order]].
- $[\forall S \subseteq \mathbb{Q}][[S \neq \emptyset][\exists s \in S][\forall q \in S][q \leq s]]$.
- QO5.** $0 < 1$ [Order of Invariance Elements [First Order]].

Second Order Axiom of "Rationality":

- Q1.** For every q in \mathbb{Q} , there is a q_z in the \mathbb{Q} -subset of \mathbb{Z} -like "Rationals", $\mathbb{Q}_z \subseteq \mathbb{Q}$, such that $q_z \times q \in \mathbb{Q}_z$ [$\subseteq \underline{\mathbb{A}\mathbb{Z}}$];
- $[\exists \mathbb{Q}_z \subseteq \mathbb{Q}][[\forall q \in \mathbb{Q}][\exists q_z \in \mathbb{Q}_z][[q_z \times q \in \mathbb{Q}_z]]]$

C.2.5. stage/step S = 5: Consolidation and ‘self-aporization’ of the System of Arithmetic of the “‘Real Numbers’”, **R**.

5.1. descriptive name of stage: *The Consolidation of the “Reals” and their “Formal Subsumption” by the “Imaginary numbers”.*

5.2. stage ‘parametrics’: [total terms #, 2^5] **32**; [new terms #, 2^{5-1}] **16**; [new terms needing solution #, $(2^{5-1} - 2)$] **14**.

5.3. «*aporia*» of this stage: $\frac{3}{H_{\#}} \mathbf{R} \dashv \oplus \frac{3}{H_{\#}} \mathbf{i}$.

5.4. “‘incompleteness’”-revealing “unsolvable” “diophantine” equation for this stage: $x_5^2 + 1 = \pm 0$; $\therefore -x_5 = +1/x_5$.

5.5. “unsolvable” equations’ paradox: *The Paradox that the Additive Inverse/Multiplicative Inverse Identity creates within* $\frac{3}{H_{\#}} \mathbf{d}$.

5.6. shortcut rendition of the ‘product-tion’ of stage 5 from stage 4 [via the ‘meta-meristematic principle’]:

$$\begin{aligned} \frac{3}{H_{\#}} \mathbf{H}_5 &= \frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{H}_4 = \\ &= \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{N} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{a} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{aN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{m} \right) \oplus \\ &= \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{mN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{ma} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{maN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{f} \right) \oplus \\ &= \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fa} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{faN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fm} \right) \oplus \\ &= \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fmN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fma} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fmaN} \right) \oplus \left(\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{d} \right) = \\ &= \frac{3}{H_{\#}} \mathbf{R} \dashv \oplus \frac{3}{H_{\#}} \mathbf{i}. \end{aligned}$$

5.7. ‘pre-solved’ new terms -- new terms of *given* or *already solved* meanings/definitions:

5.7.1. $\equiv \left(\frac{3}{H_{\#}} \mathbf{r} \right) \mathbf{E} \Rightarrow \mathbb{Q}_{31}$ -- first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’

of the “‘epitome’” of the “Reals”, $\mathbf{r} \equiv \{ (+0...01.0...), (+0...02.0...), (+0...03.0...), \dots \}$, the sub-space of the **R**

number-space which expresses the fruition of the *critique* of $\frac{3}{H_{\#}} \mathbf{fmaN}$ by $\frac{3}{H_{\#}} \mathbf{d}$ in the product $\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{fmaN} =$

$\frac{3}{H_{\#}} \mathbf{d} \otimes \frac{3}{H_{\#}} \mathbf{f}$, namely $\frac{3}{H_{\#}} \mathbf{dfmaN} = \frac{3}{H_{\#}} \mathbf{r}$, the sub-space of the **R** number-space containing all $\frac{3}{H_{\#}} \mathbf{d}$ -converted

$\frac{3}{H_{\#}} \mathbf{f}$ -converted $\frac{3}{H_{\#}} \mathbf{m}$ -converted $\frac{3}{H_{\#}} \mathbf{a}$ -converted “Natural” Numbers, $\mathbf{N} \equiv \{ \mathbf{I}, \mathbf{II}, \mathbf{III}, \dots \}$.

5.7.2. $\equiv \left(\frac{3}{H_{\#}} \mathbf{i} \right) \mathbf{E} \Rightarrow \mathbb{Q}_{32}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘inverses-identity-based’ Numbers, or of the so-called “imaginary numbers” $| \mathbf{i}^2 \frac{1}{\mathbf{i}} \& -\mathbf{i} = +1/\mathbf{i}$;

$\mathbf{i} \equiv \{ \mathbf{di} \} \equiv \{ +\sqrt{-d^2} \mid \pm 0 < d \in \mathbf{d} \subset \mathbf{R} \}$, i.e., the totality of *positive* square-roots of all ‘negativized **d**agonal numbers’, $-\mathbf{d}$;

fruition of the [immanent] *self-critique* of $\frac{3}{H_{\#}} \mathbf{d}$, namely $\frac{3}{H_{\#}} \mathbf{dd} = \frac{3}{H_{\#}} \mathbf{i}$.

5.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module ζ]:

5.8.1. $\equiv \left(\overset{3}{\underset{\#}{\mathcal{H}}} \overset{\#}{\mathcal{Q}} \right) \overset{\#}{\mathcal{E}} \rightarrow \left[\overset{\#}{\mathcal{Q}} \right]_{17}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘*decimal-ized*’ “*N*atural” Numbers, $\mathbf{dN} \equiv \{ \mathbf{I....}, \mathbf{II....}, \mathbf{III....}, \dots \}$, the sub-space of the *R* number-space which expresses the fruition of the *critique* of $\overset{3}{\underset{\#}{\mathcal{H}}} \mathbf{N}_{\#}$ by $\overset{3}{\underset{\#}{\mathcal{H}}} \mathbf{d}_{\#}$ in the product $\overset{3}{\underset{\#}{\mathcal{H}}} \mathbf{d}_{\#} \otimes \overset{3}{\underset{\#}{\mathcal{H}}} \overset{\#}{\mathcal{Q}}_{\mathbf{N}}$, namely $\overset{3}{\underset{\#}{\mathcal{H}}} \overset{\#}{\mathcal{Q}}_{\mathbf{dN}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment...’” of $\overset{3}{\underset{\#}{\mathcal{H}}} \mathbf{N}_{\#}$ with/with/by/by/to... $\overset{3}{\underset{\#}{\mathcal{H}}} \mathbf{d}_{\#}$ as the elevation of $\overset{3}{\underset{\#}{\mathcal{H}}} \mathbf{N}_{\#}$ into *R*.

5.8.2. $\equiv \left(\begin{matrix} 3 \\ H \end{matrix} \right)_{\#}^{\text{da}} \rightarrow \left(\begin{matrix} 3 \\ H \end{matrix} \right)_{18}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘*decimal-ized*’ ‘*aught*’ Numbers, $\text{da} \equiv \{ (\text{I} - \text{I}).0..., (\text{II} - \text{II}).0..., (\text{III} - \text{III}).0..., ... \}$, the sub-space of the \mathbf{R} number-space which expresses the fruition of the *critique* of $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{a}}$ by $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{d}}$ in the product $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{d}} \otimes \begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{a}}$, namely $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{da}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment...” of $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{a}}$ with/by/to... $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{d}}$ as $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{a}}$ ’s elevation into \mathbf{R} .

5.8.3. $\equiv \left(\overset{3}{\underset{\text{daN}}{\mathbb{H}}} \overset{\#}{\mathbb{Q}} \right) \overset{\#}{\mathbb{E}} \rightarrow \left[\right]_{19}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘decimal-ized’ ‘aught-ized Natural Numbers’, $\text{daN} \equiv \{ (0...01.0...), (0...02.0...), (0...03.0...), \dots \}$, the sub-space of R number-space which expresses the fruition of the critique of $\overset{3}{\underset{\text{aN}}{\mathbb{H}}} \overset{\#}{\mathbb{Q}}$ by $\overset{3}{\underset{\#}{\mathbb{H}}} \overset{\#}{\mathbb{d}}$ in the product $\overset{3}{\underset{\#}{\mathbb{H}}} \overset{\#}{\mathbb{d}} \otimes \overset{3}{\underset{\text{aN}}{\mathbb{H}}} \overset{\#}{\mathbb{Q}}$, namely $\overset{3}{\underset{\text{daN}}{\mathbb{H}}} \overset{\#}{\mathbb{Q}}$, which is also the “‘hybridization/synthesis/subsumption...’” of $\overset{3}{\underset{\#}{\mathbb{H}}} \overset{\#}{\mathbb{a}}$ with/with/by... $\overset{3}{\underset{\#}{\mathbb{H}}} \overset{\#}{\mathbb{d}}$ as its elevation into R.

5.8.4. $\equiv \left(\begin{matrix} 3 \\ H \end{matrix} \right)_{\#}^{\circledast} \rightarrow \left[\begin{matrix} 20 \end{matrix} \right]_{\square}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’ of the decimal-ized ‘minus’ Numbers, $\text{dm} \equiv \{ (0-1)\dots, (0-2)\dots, (0-3)\dots, \dots \}$, the sub-space of the R number-space which expresses the fruition of the critique of $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{m}}$ by $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{d}}$ in the product $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\text{d}} \otimes \begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\circledast} \text{m}$, namely $\begin{pmatrix} 3 \\ H \end{pmatrix}_{\#}^{\circledast} \text{dm}$.

5.8.5. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{dmN}} \right) \right) \rightarrow \left(\overset{21}{\text{L}} \right)$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’ of ‘decimal-ized’ ‘sign-ized’ Natural Numbers’, $\text{dmN} \equiv \{+1\text{....}, +2\text{....}, +3\text{....}, \dots\}$, the sub-space of R number-space which expresses the fruition of the critique of $\left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{mN}} \right) \right)$ by $\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{d}} \right)$ in the product $\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{d}} \right) \otimes \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{mN}} \right) \right)$, namely $\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{dmN}} \right)$.

5.8.6. $\equiv \left(\begin{matrix} 3 \\ H \\ \textcircled{\#} \end{matrix} \right)_{\text{dma}}^{\#} \xrightarrow{\quad} \boxed{22}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘*d*ecimal-ized’ ‘*s*ign-ized *a*ught Numbers’, $\text{dma} \equiv \{ \pm(I - I).0... = (\pm 0.0...), \pm(II - II).0... = (\pm 0.0...), ... \}$, the sub-space of **R** number-space which expresses the fruition of the *critique* of $\begin{matrix} 3 \\ H \\ \textcircled{\#} \end{matrix}_{\text{ma}}$ by $\begin{matrix} 3 \\ H \\ \underline{\#} \end{matrix}_{\text{d}}$ in the product $\begin{matrix} 3 \\ H \\ \underline{\#} \end{matrix}_{\text{d}} \otimes \begin{matrix} 3 \\ H \\ \textcircled{\#} \end{matrix}_{\text{ma}}$, namely $\begin{matrix} 3 \\ H \\ \textcircled{\#} \end{matrix}_{\text{dma}}^{\#}$, which is also the “‘hybridization/synthesis/subsumption...”” of $\begin{matrix} 3 \\ H \\ \underline{\#} \end{matrix}_{\text{a}}$ with/by... $\begin{matrix} 3 \\ H \\ \underline{\#} \end{matrix}_{\text{d}}$ as its elevation into **R**.

5.8.7. $\equiv \left(\left({}^3_H\text{Q} \right)^{\#} \right) \mapsto \boxed{\text{J}_{23}}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘decimal-ized’ ‘sign-ized’ ‘aught-ized’ “Natural Numbers”, $\mathbf{dman} \equiv \{(\pm 0...01.0...), (\pm 0...02.0...), \dots\}$,

the sub-space of **R** number-space which expresses fruition for the critique of $\overset{3}{\text{H}}\overset{\#}{\text{g}}\text{maN}$ by $\overset{3}{\text{H}}\overset{\#}{\text{d}}$ in the product $\overset{3}{\text{H}}\overset{\#}{\text{d}} \otimes \overset{3}{\text{H}}\overset{\#}{\text{g}}\text{maN}$,
namely $\overset{3}{\text{H}}\overset{\#}{\text{g}}\text{dman}$, which is also the “‘hybridization/synthesis/subsumption...’” of $\overset{3}{\text{H}}\overset{\#}{\text{a}}$ with/with/by... $\overset{3}{\text{H}}\overset{\#}{\text{d}}$ as its elevation into **R**.

5.8.8. $\equiv \left(\left(\overset{3}{H} \right) \left(\overset{\#}{\text{df}} \right) \right) \mapsto \boxed{24}$ -- the *first-and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the 'decimal-ized' fractional Numbers', $\mathbf{df} \equiv \{ \mathbf{d}_f = (f_1/f_2)_{\dots} \mid 0 \neq f_2, \& f_1, f_2 \in \mathbf{f} \}$, the sub-space of the \mathbf{R}

number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\text{f}}{\underset{\#}{\text{f}}}$ by $\overset{3}{\text{H}}\overset{\text{d}}{\underset{\#}{\text{f}}}$ in the product $\overset{3}{\text{H}}\overset{\text{d}}{\underset{\#}{\text{f}}} \otimes \overset{3}{\text{H}}\overset{\text{f}}{\underset{\#}{\text{f}}}$, namely $\overset{3}{\text{H}}\overset{\text{f}}{\underset{\#}{\text{df}}}$.

5.8.9. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{Q}} \right) \right) \text{dfN} \text{E} \Rightarrow \left(\text{I} \right)_{25}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of 'decimal-ized' 'fraction-ized' "Natural Numbers", $\text{dfN} \equiv \{ (\text{I/I})_{\dots}, (\text{II/I})_{\dots}, (\text{III/I})_{\dots}, \dots \}$, the sub-space of **R**

number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\#}{\text{fN}}$ by $\overset{3}{\text{H}}\overset{\text{d}}{\text{d}}_{\#}$ in the product $\overset{3}{\text{H}}\overset{\text{d}}{\text{d}}_{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\#}{\text{fN}}$, namely $\overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\text{dfN}}{\text{d}}_{\#}$.

5.8.10. $\equiv \langle \langle \overset{3}{H} \overset{\#}{Q} \rangle \rangle_{\text{dfa}} \mapsto \boxed{26}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of 'decimal-ized' 'fraction-ized' 'ought Numbers', $\mathbf{dfa} \equiv \{ ((\mathbf{I} - \mathbf{I})/(\mathbf{I}))...., ((\mathbf{II} - \mathbf{II})/(\mathbf{I}))...., ((\mathbf{III} - \mathbf{III})/(\mathbf{I}))...., \dots \}$,

the sub-space of the **R** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{fa}}$ by $\overset{3}{\text{H}}\overset{\text{d}}{\text{fa}}$ in the product $\overset{3}{\text{H}}\overset{\text{d}}{\text{fa}} \otimes \overset{3}{\text{H}}\overset{\#}{\text{fa}}$, namely $\overset{3}{\text{H}}\overset{\text{d}}{\text{fa}}$, which is also the “hybridization/synthesis/subsumption...” of $\overset{3}{\text{H}}\overset{\#}{\text{fa}}$ with/with/by... $\overset{3}{\text{H}}\overset{\text{d}}{\text{fa}}$ as its elevation into **R**.

5.8.11. $\equiv \left(\left(\overset{3}{\underset{\text{dfaN}}{\text{H}}} \right)^{\#} \right) \mapsto \mathbb{T}_{27}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the '**d**ecimal-ized' '**f**raction-ized' '**a**ught-ized' "N**a**tural Numbers", **d**fa**N** $\equiv \{ (0...01/0...01)_{...}, \dots \}$, the

sub-space of the **R** number-space which expresses the fruition of the critique of $\text{H} \begin{smallmatrix} \text{g} \\ \text{faN} \end{smallmatrix}^\#$ by $\text{H} \begin{smallmatrix} \text{d} \\ \# \end{smallmatrix}$ in the product $\text{H} \begin{smallmatrix} \text{d} \\ \# \end{smallmatrix} \otimes \text{H} \begin{smallmatrix} \text{g} \\ \text{faN} \end{smallmatrix}^\#$,
namely $\text{H} \begin{smallmatrix} \text{g} \\ \text{dfaN} \end{smallmatrix}^\#$, which is also the ““hybridization/synthesis”” of $\text{H} \begin{smallmatrix} \text{g} \\ \text{faN} \end{smallmatrix}^\#$ with $\text{H} \begin{smallmatrix} \text{d} \\ \# \end{smallmatrix}$ as the elevation of $\text{H} \begin{smallmatrix} \text{g} \\ \text{faN} \end{smallmatrix}^\#$ into **R**.

5.8.12. $\equiv \left(\left(\overset{3}{H} \overset{\#}{\bigcirc} \right) \right) \mapsto \boxed{28}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

'sub-arithmetic' of 'decimal-ized' 'fraction-ized' 'minus numbers', $\mathbf{dfm} \equiv \{ ((0-1)/+1)_{\dots}, ((0-2)/+1)_{\dots}, ((0-3)/+1)_{\dots}, \dots \}$, the

sub-space of **R** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{d}}\overset{\text{fm}}{\text{f}}$ by $\overset{3}{\text{H}}\overset{\text{d}}{\text{f}}\overset{\#}{\text{m}}$ in the product $\overset{3}{\text{H}}\overset{\text{d}}{\text{f}}\overset{\#}{\text{m}} \otimes \overset{3}{\text{H}}\overset{\text{d}}{\text{f}}\overset{\#}{\text{m}}$,
namely $\overset{3}{\text{H}}\overset{\text{d}}{\text{f}}\overset{\text{fm}}{\text{f}}\overset{\#}{\text{m}}$, which is also the ““hybridization/synthesis/subsumption...”” of $\overset{3}{\text{H}}\overset{\text{d}}{\text{f}}\overset{\text{fm}}{\text{f}}\overset{\#}{\text{m}}$ with/with/by... $\overset{3}{\text{H}}\overset{\text{d}}{\text{f}}\overset{\#}{\text{m}}$ as its elevation to **R**.

5.8.13. $\equiv \left(\text{H}^{\text{3}}_{\text{Q}} \right)^{\#} \text{E} \rightarrow \boxed{}_{29}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of ‘decimal-ized’ ‘fraction-ized’ ‘sign-ized “Naturals”, $\mathbf{dfmN} \equiv \{ (+1/+1)_{\dots}, (+2/+1)_{\dots}, (+3/+1)_{\dots}, \dots \}$, the

sub-space of **R** number-space which expresses the fruition of the critique of $\overset{\#}{\underset{\text{fmN}}{\text{H}}}\text{d}$ by $\overset{\#}{\underset{\text{d}}{\text{H}}}$ in the product $\overset{\#}{\underset{\text{d}}{\text{H}}}\otimes\overset{\#}{\underset{\text{fmN}}{\text{H}}}$, namely $\overset{\#}{\underset{\text{dfmN}}{\text{H}}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\overset{\#}{\underset{\text{fmN}}{\text{H}}}$ with/with/by/by/to/and $\overset{\#}{\underset{\text{d}}{\text{H}}}$ as the elevation of $\overset{\#}{\underset{\text{fmN}}{\text{H}}}$ up into **R**.

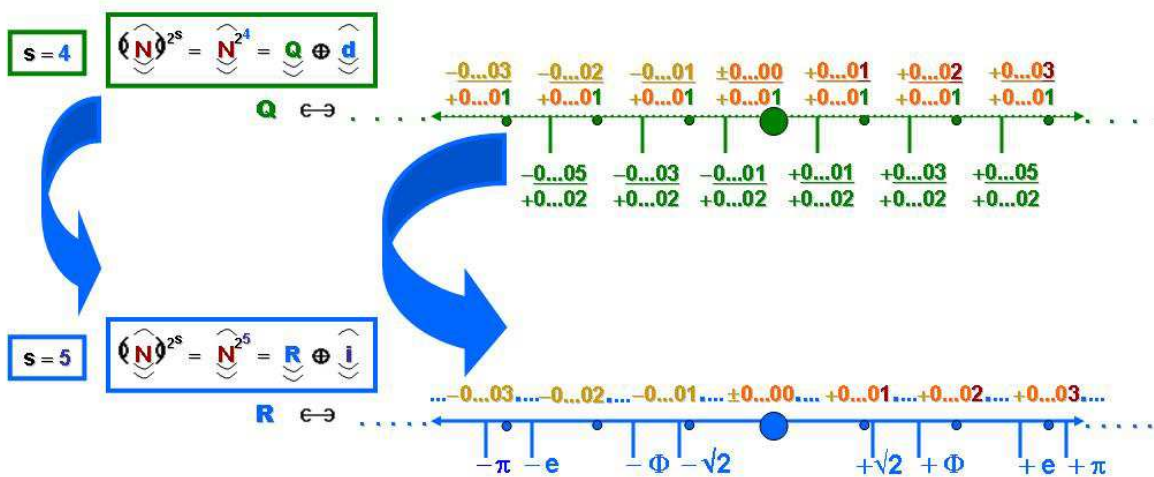
5.8.14. $\equiv \left(\left(\overset{3}{\underset{\text{H}}{\text{O}}} \right) \overset{\#}{\text{d}} \right) \left[\text{E} \rightarrow \left(\left[\text{ } \right]_{30} \right) \right]$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the 'decimal-ized' 'fraction-ized' 'sign-ized ought Numbers', $\mathbf{dfma} \equiv \{ (\pm(\mathbf{I} - \mathbf{I})/(\mathbf{+1}))_{\dots} = (\pm\mathbf{0}/\mathbf{+1})_{\dots}, \dots \}$, the

sub-space of the **R** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{fma}$ by $\overset{3}{\text{H}}\text{d}_{\#}$ in the product $\overset{3}{\text{H}}\text{d}_{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{fma}$, namely $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{dfma}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{fma}$ with/with/by/by/to/and $\overset{3}{\text{H}}\text{d}_{\#}$ as the elevation of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{fma}$ up into **R**.

Number-Line Cumulative Progression Analytic-“Geometric” Model
of the **Gödelian Dialectic of the Standard Arithmetics:**

The Transition from $s = 4$ to $s = 5$, and from $Q \oplus \dots$ to $R \oplus \dots$



$3H \oplus dN$ $3H \oplus da$ $3H \oplus daN$ $3H \oplus dm$ $3H \oplus dmN$ $3H \oplus dma$ $3H \oplus dmaN$ $3H \oplus df$
 $3H \oplus dfN$ $3H \oplus dfa$ $3H \oplus dfaN$ $3H \oplus dfm$ $3H \oplus dfmN$ $3H \oplus dfma$ $3H \oplus dfmaN$

$3\text{H}^{\text{R}}_{\sharp} \equiv 3\text{H}^{\text{G}}_{\sharp} \oplus 3\text{H}^{\text{d}}_{\sharp} \oplus 3\text{H}^{\text{g}}_{\sharp} \oplus 3\text{H}^{\text{dN}}_{\sharp} \oplus 3\text{H}^{\text{da}}_{\sharp} \oplus 3\text{H}^{\text{daN}}_{\sharp} \oplus 3\text{H}^{\text{dm}}_{\sharp} \oplus 3\text{H}^{\text{dmN}}_{\sharp} \oplus 3\text{H}^{\text{dma}}_{\sharp} \oplus 3\text{H}^{\text{dmaN}}_{\sharp}$
 $\oplus 3\text{H}^{\text{df}}_{\sharp} \oplus 3\text{H}^{\text{dfN}}_{\sharp} \oplus 3\text{H}^{\text{dfa}}_{\sharp} \oplus 3\text{H}^{\text{dfaN}}_{\sharp} \oplus 3\text{H}^{\text{dfm}}_{\sharp} \oplus 3\text{H}^{\text{dfmN}}_{\sharp} \oplus 3\text{H}^{\text{dfma}}_{\sharp} \oplus 3\text{H}^{\text{dfmaN}}_{\sharp}$

$$\Delta \left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \begin{smallmatrix} d \\ \# \end{smallmatrix} \right) \equiv \begin{smallmatrix} 3 \\ H \end{smallmatrix} \begin{smallmatrix} \# \\ d \end{smallmatrix}$$

whole “*genre*” of number “*Real-ity*”? ³More specifically, is there a kind-of-number(s) that is *opposite* to the ³H_d# kind-of-number(s), that corresponds to the fruition

Systems, is one dominated by the $\mathbf{3}_{\mathbf{Hd}} \ll \mathbf{aufheben} \gg$ operator, through its operation upon / interaction with, and via its “‘synthesis’” with / “‘hybridization’” with / ‘complex unification’ with / “‘**Real subsumption**’” of, every arithmetical axioms-system *component*/kind-of-number(s) *term* that was already evoked in all previous

$\text{H}_{\#}^{\underline{\mathbf{d}}}\otimes \text{H}_{\#}^{\underline{\mathbf{d}}}=\text{H}_{\#}^{\underline{\mathbf{d}}}\oplus \text{H}_{\#}^{\underline{\mathbf{d}}}= \text{H}_{\#}^{\underline{\mathbf{d}}}\oplus \text{H}_{\#}^{\underline{\mathbf{d}}}\left(\text{H}_{\#}^{\underline{\mathbf{d}}}\right)$. This ‘d-ization’ of the \mathbf{Q} --the addition, to the incipient number “line” of the $\mathbf{s}=4$

“continuous” and seemingly “solid”, *replete* with ‘number-points’, and corresponding to the **3H_#** system -- has worked a further transformation in point-of-view,

Vignette #4 *The Gödelian Dialectic of the Standard Arithmetics* **4.4.0.II -39** by M. Detonacciones, **Foundation Encyclopedia Dialectica** [F.E.D.]

We might thus suspect that the **S = 5 stage**'s emergent 'self-opposition' of/to/within the $\frac{3}{H}R_{\#}$ system might involve an internal breach, or 'immanent transcendence', of these very 'number-definitional' deficits.

The $\frac{3}{H}R_{\#}$ "**Reals**" are, each and all, 'potentially-infinite decimalizations', expressible as the **decimal values**, ending in ellipsis, with **repeating** digits, or **repeating sequences** of digits, for "**ratio-nal real numbers**", with **repeating zeros** for "**integ[er]al real numbers**", and with tendentially **never-repeating digit-sequences** for "**ir-ratio-nal real numbers**". The **R** number-space is a 'unitation' of the "**rational**", **Q**, of the '**ir-rational**' subset **d**, and of the 'potentially-infinite decimalizations' of every "inherited" component of the **Q** space. The arithmetic system $\frac{3}{H}R_{\#}$ is $\frac{3}{H}d_{\#} \otimes \frac{3}{H}Q_{\#}$, the ' $\frac{3}{H}d_{\#}$ -ization', or **«aufheben»** ' $\frac{3}{H}d_{\#}$ -elevation', of $\frac{3}{H}Q_{\#}$, in all of its components, component-by-component, and which already includes the "evolute", [double-]**«aufheben» conservation** of $\frac{3}{H}Q_{\#}$.

The **supplementary-opposite** kind-of-number(s) to the **R** kind, would therefore evidently be a [sub-]system-category of '**non-«Real» numbers**' **in some sense**.

We identify/interpret these '**non-Real numbers**', with/as the [sub-]system of the so-called '**I**maginary numbers', that we have already found extant in our perhaps unsystematic, "chaotic" [Marx] knowledge/experience of the totality of the modern/contemporary 'Standard Numbers': the kind of number(s) that we denote herein by:

$$\frac{3}{H}i_{\#} \mapsto \frac{3}{H}i_{\#}$$

We have thus solved for the meaning of the '**contra-thesis**' category of the **d** subset of the "**irrational numbers**" '**contra-thesis**' category **itself** --

$$\Delta\left(\frac{3}{H}d_{\#}\right) = \frac{3}{H}i_{\#} \text{ as } \frac{3}{H}i_{\#}; \quad \Delta\left(\frac{3}{H}d_{\#}\right) = \frac{3}{H}i_{\#}$$

We call the ' $\frac{3}{H}i_{\#}$ numbers' the '**I**maginary numbers', for [psycho]historical reasons, because, at the time of their original naming, e.g., square roots of **negative «Real» numbers** seemed **impossible, inconceivable**, to the mathematical, arithmetical-conceptual horizon of the *then-existing, then-contemporary, then-consensus* human phenome, such that such square-roots could not be "**real**" -- i.e. "**true**" -- **numbers**. Yet, because of their manifest usefulness, such **values** were 'rule-ified' and admitted under the intentionally derogatory label of "**imaginary**" -- i.e., "**illusory**", "**un-real**", "**false**", or "**fictitious**" -- numbers. A truer epithet for these **values** might be '**counter-Real**'.

Strictly in terms of our "analytic-geometric" vision/visualization of the Standard Arithmetics, first addressed in section **B.η.**, the emergence of the **a** from out of the self-critique of the **N** filled in a slot to the left of all of the "**N**atural" Numbers "points", and the fruition of the **self-critique** of $\frac{3}{H}a_{\#}$ as $\frac{3}{H}m_{\#}$, marked a "filling-in" of a potentially-infinite 'counter-ray' of 'number-"points"', again all to the left of all of the "**N**atural" Numbers "points". These first two movements, from the **N** to the **W**, and then from the **W** to the **Z**, both involve an "**extensive filling-in**" -- in the form of a "filling-out[ward]" -- of number-space. The emergence of the **f** from out of the self-critique of the **m** marked the onset of a new phase; one also consisting of two consecutive movements -- first, from the **Z** to the **Q**, and, second, from the **Q** to the **R**, the latter having the **d** as their *limen* -- a phase of "**intensive filling-in**", in which new 'number-"points"' are added-in, distributed amongst the 'number-"points"' previously 'extantized' explicitly, or 'outed', for the **Z**, and, then, next, even more so, for the **Q**. In particular, the transition from the **Q** to the **R**, mediated and effected by $\frac{3}{H}d_{\#}$, packs new 'number-"points"' "**continuously**", **in-between** every pair of those '**Q whole-number-"points"**', '**whole-number** "points"' that have nothing other than a potentially-infinite succession of **0s** to the right of their **decimal points** in their **decimal representations**. The harbinger of this transformation, $\frac{3}{H}d_{\#}$, in itself, merely posits this "**continuity**" for the **RHS**, 'positive-signed ray' of the **Q-numbers** "line"-space.

But the advent of the **d**, nevertheless, already portends the comprehensive '**continuity**' and '**potentially-infinite decimal-ization**' of this single "number-line" entire, negative, neutral, and positive alike, that arrives in the next **stage/step** of our 'meta-model', **stage/step S = 5**. In any case, the only at last '**numbers-dense**' -- but all along and still vastly 'por[e]-ous' -- '**pseudo-solid**' "line" of numbers that pertained from the **N** to the **W** to the **Z** and, at last, even to the **Q**, is, already, "here", in

stage/step S = 4, even with its merely "**formal subsumption**" of $\frac{3}{H}Q_{\#}$ by $\frac{3}{H}d_{\#}$, **on the liminal verge** of becoming a "**Real**", "**solid**", "**line**" -- a "**continuous**",

rectilinear, **1-D** expanse "of [number-]points", **without any more lacunae**, at least not from the $\frac{3}{H}d_{\#}$ viewpoint -- or from the viewpoint of the "**Standard**" $\frac{3}{H}R_{\#}$.

With the advent of the **i** increment of number 'ideo-ontology', a radically new mode of continuance of this continuing process of the "filling-in" of "number-space" with ever new **kinds** of 'number-points' is born. The progressive "filling-in" of a single "number-line" 'number-dimension', all the way up to the point of "**solidity**", suddenly "morphs" into the addition, whole, of an entire new, equally "**solid**"/'**continuous**', **second** "number-line" 'number-dimension', perpendicular to the first.

The Equation, “‘Well-Formed’” within $\frac{3}{H}R_{\#}$, that is *Not Solvable* within $\frac{3}{H}R_{\#}$, and that Therefore Points, from within $\frac{3}{H}R_{\#}$, to beyond $\frac{3}{H}R_{\#}$ -- to within $\frac{3}{H}C_{\#}$.

In terms of an immanent critique, or self-critique, of the “*Reals*” system of arithmetic, and, more specifically, in terms of the self-critique of the “*diagonal*” numbers [sub-]system of arithmetic, we have noted a family of algebraic equations, “‘well-formed’” within the algebra of the “*Reals*” arithmetic, which are, however, *not* “satisfiable” within the “*Reals*” system -- which are *not* “solvable” by any “*Real*” number. This ‘equations-family’ can be represented, generically, by the single equation, $[x_{\epsilon}^2 = -d, 0 < d \in R]$. A specific instance of this equations-family is $x_{\epsilon}^2 = -1$, i.e., $x_{\epsilon} = \pm\sqrt{-1}$. This equation, not by immediate inspection, but upon further algebraical analysis, is found to implicitly assert a paradox for a definition of ‘Standard Number’ restricted to the concept of the “*Reals*”: ‘*the paradox of additive inverse/multiplicative inverse identity*’ -- $[x_{\epsilon}^2 = -1] \Rightarrow [x_{\epsilon}^2/x_{\epsilon}^1 = -1/x_{\epsilon}^1] \Rightarrow [x_{\epsilon}^1 = -1/x_{\epsilon}] \Rightarrow [-x_{\epsilon} = +1/x_{\epsilon}]$.

Therefore, the self-critique of the “*Reals*”, as seeded by the equation analyzed above, and, more specifically, the self-critique of ‘the *diagonal* numbers’, leads us to the ‘*imaginary numbers*’, $\frac{3}{H}i_{\#} \stackrel{\#}{=} \frac{3}{H}i_{\#}$, the category of *numbers* that adds a whole new, equally “*continuous*” dimension to our developing “number-space”.

The *i* kind-of-numbers, in this *stage*’s now “‘*formal subsumption*’” of/contrast with the *d* kind & their [sub-]system of arithmetic, $\frac{3}{H}d_{\#}$, and their fullness as the *R* kind-of-number(s), and their system of arithmetic, $\frac{3}{H}R_{\#}$, and in terms of our “‘analytic geometric’” visualization of these “number-spaces”, redefine ‘Standard Number’ to be, not just “‘counts’”, or at least ‘a-mounts’, in either of the two directions/“rays” inherent in a single “line”, a single “dimension” of number-kind, but also “‘counts’”, or at least ‘a-mounts’, along either of the two directions/“rays” inherent in a second “line”, a second “dimension”, of number-kind, orthogonal to the first.

This *stage* of our ‘self-dialogue’ has therefore far *far* further “filled-in” our “analytic-geometric” number-space. It has equipped us conceptually with the inkling of a whole *new second “fully-line-ar” number-space*, right after -- upon the very heels of -- our apparent completion of *the first “fully-line-ar” number space*, and thus, given combinations/interactions between the two [sub-]spaces, more than doubling the *content* of our developing “number-space”. Via the advent of $\frac{3}{H}i_{\#}$, the domain of number is now ‘re-seen’, in *this new relative light*, as *potentially multi-dimensional* -- as at least two-dimensional -- rather than as *merely one-dimensional*. Thus, the *five* preceding horizons of our number-concept -- namely, those of the *N*, the *W*, the *Z*, the *Q*, and the *R*, are, together, [re-]visualized, in the perspective opened via the vantage-point of $\frac{3}{H}i_{\#}$, as but *five* successive *stages* in the construction of merely *the first dimension* of an at least *two-dimensional space* of number-kinds.

Taking the Measure of Where We Have Arrived Cognitively via Step/Stage 5 in this Chain of Immanent Critiques.

Thus, we arrive at -- as the fruition of *Step/Stage 5* -- the ‘mutual controversion’ of $\frac{3}{H}R_{\#}$ vs. $\frac{3}{H}i_{\#}$. We have $\frac{3}{H}R_{\#}$ as the partial, one-sided, unnecessarily restrictive view of modern ‘Standard Number’ as limited to “‘*continuous-counting*’”, and to the “‘*non-counting*’” of, and in, the *units of the square root of negative “real” unity*. And we have this $\frac{3}{H}R_{\#}$, in an ‘additively oppositional’ relation to $\frac{3}{H}i_{\#}$ as *counter-example* to $\frac{3}{H}R_{\#}$ ’s ‘number as *continuous-parts-of-units-counting, square roots of negative “real” units-ignoring*’ view. That is, we have $\frac{3}{H}i_{\#}$ as a [sub-]system of ‘*perpendicularly-transcending-Real* number-“points”’.

However, in itself, $\frac{3}{H}i_{\#}$ is just as “one-sided”/“partial” -- just as incomplete/inadequate -- as $\frac{3}{H}R_{\#}$, if not even more so. The $\frac{3}{H}i_{\#}$ [sub-]system of arithmetic encompasses *only i-unit*-based, *continuous-parts-of-i-units-counting* numbers, excluding/ignoring the entire *r-unit*-ed, *R[real] dimension* of number-kind.

So $\frac{3}{H}R_{\#}$ is equally a *counter-example* to $\frac{3}{H}i_{\#}$ -- to any claims of adequacy/completeness/totality that might be ‘mis-made’ on behalf of $\frac{3}{H}i_{\#}$.

We thus remain in an unsatisfactory situation, in terms of systematically -- including of “‘taxonomically’” -- organizing & accounting for our entire experience of the totality of modern, contemporary “Standard Arithmetic(s)”, although the ‘explicitization’ of $\frac{3}{H}i_{\#}$ does mark a massive advance beyond the even more unsatisfactory situation wherein we dwelt in *Step/Stage 4*, and, even more so, wherein we dwelt in the *Steps/stages before Step/Stage 4*.

But to improve our situation further, we must move on, on to *Step/Stage 6*, to $\frac{3}{H}H_{\#}^{(6)} = \left(\frac{3}{H}R_{\#}\right)^2$, to the *self-«aufheben»*, i.e., to the *self-reflection*, or

“‘*squaring with itself*’”, of *Step/Stage 5*’s own result, namely, of $\left(\frac{3}{H}R_{\#} \dashv \frac{3}{H}i_{\#}\right)$ [with] itself, or [shortcut] to the results of the action of ‘[de]flection’ of the collective «*aufheben*» operator/operation denoted $\left(\frac{3}{H}R_{\#} \dashv \frac{3}{H}i_{\#}\right)$ by the «*aufheben*» operator/operation denoted $\frac{3}{H}i_{\#}$.

What the Symbol ${}^3_H\mathbf{R}_\#$ "Intends": *Axioms-System of the Arithmetic* of the so-called "**Real**" Numbers, \mathbf{R} [commenced]

\mathbf{R}, \mathbf{R} is not the Empty Set. $\mathbf{R} \neq \emptyset \equiv \{ \}$;

First Order *Peano's* Axioms: (mainly) "Phonogramic" Rendering: (mainly) "Ideogramic" Rendering

RP1. $\mathbf{0}$ is a "Real" Number. $\mathbf{0} \in \mathbf{R}$

RP2. The **successor** of a "Real" Number is also a "Real" Number. $r \in \mathbf{R} \Rightarrow S(r) \in \mathbf{R}$

RP3. No two "Real" Numbers have the same **successor**. $\neg \exists r_1, r_2 \in \mathbf{R} [r_1 \neq r_2 \wedge S(r_1) = S(r_2)]$

RP4. $\neg [\mathbf{0}$ is **not** the **successor** of any other "Real" Number]. $\neg \exists x \in \mathbf{R} [S(x) = \mathbf{0} \wedge x = -1]$ *the predecessor in \mathbf{R} , and that predecessor is -1*

Second Order *Mathematical Induction* Axiom for "Real" Arithmetic for Set S as *Extension of Intension* (Predicate) P

RP5. $\forall r \in \mathbf{R} [r \in S] \wedge [r+1 \in S] \Rightarrow [r-1 \in S] \Rightarrow [S = \mathbf{R}]$ [Induction for the "Reals"]

First Order Axioms of "Real" Numbers **Addition**:

RA1. For all r_1, r_2 in \mathbf{R} , $r_1 + r_2 = r_2 + r_1$ [Additive Commutativity];

RA2. For all r_1, r_2, r_3 in \mathbf{R} , $(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$ [Additive Associativity];

RA3. For all r_1, r_2 in \mathbf{R} , $r_1 + r_2$ is in \mathbf{R} [Additive Closure of \mathbf{R}]; $[\forall r_1, r_2 \in \mathbf{R}] [r_1 + r_2 \in \mathbf{R}]$;

RA4. There is an element $\mathbf{0}$ in \mathbf{R} such that, for all r in \mathbf{R} , $\mathbf{0} + r = r$ [Additive Invariance Element];

RA5. For every r in \mathbf{R} , there exists $(-r)$ in \mathbf{R} , such that $r + (-r) = \mathbf{0}$ [Additive Inverse Elements];

$$[\forall r \in \mathbf{R}] [\exists -r \in \mathbf{R}] [r + (-r) = (-r) + r = \mathbf{0}];$$

First Order Axioms of "Real" Numbers **Multiplication**:

RM0. There is an element $\mathbf{0}$ in \mathbf{R} such that, for all r in \mathbf{R} , $\mathbf{0} \times r = \mathbf{0}$ [Multiplicative Predominance];

$$[\exists \mathbf{0} \in \mathbf{R}] [\forall r \in \mathbf{R}] [\mathbf{0} \times r = r \times \mathbf{0} = \mathbf{0}];$$

RM1. For all r_1, r_2 in \mathbf{R} , $r_1 \times r_2 = r_2 \times r_1$ [Multiplicative Commutativity];

RM2. For all r_1, r_2, r_3 in \mathbf{R} , $(r_1 \times r_2) \times r_3 = r_1 \times (r_2 \times r_3)$ [Multiplicative Associativity];

RM3. For all r_1, r_2 in \mathbf{R} , $r_1 \times r_2$ is in \mathbf{R} [Multiplicative Closure of \mathbf{R}]; $[\forall r_1, r_2 \in \mathbf{R}] [r_1 \times r_2 \in \mathbf{R}]$;

RM4. There is an element $\mathbf{1}$ in \mathbf{R} such that, for all r in \mathbf{R} , $\mathbf{1} \times r = r$ [Multiplicative Invariance Element];

RM5. For every $r \neq \mathbf{0}$ in \mathbf{R} , there exists $(1/r)$ in \mathbf{R} , such that $r \times (1/r) = \mathbf{1}$ [Multiplicative Inverse Elements];

$$[\forall r \in \mathbf{R}] [\exists 1/r \in \mathbf{R}] [r \times (1/r) = (1/r) \times r = \mathbf{1}]; \text{ the value of the expression } (r/\mathbf{0}) \text{ is undefined;}$$

What the Symbol ${}^3_H\mathbf{R}_\#$ "Intends" [continued & concluded].

First Order Axioms for the "Hybridization" of the "Real" Numbers Operations of "Multiplication" and of "Addition":

RH1. For all r_1, r_2, r_3 in \mathbf{R} , $r_1 \times (r_2 + r_3) = r_1 \times r_2 + r_1 \times r_3$ [Distributivity ["Hybridization"] of Multiplication over Addition];

RH2. For all r_1, r_2, r_3 in \mathbf{R} , $(r_1 + r_2) \times r_3 = r_1 \times r_3 + r_2 \times r_3$ [Distributivity ["Hybridization"] of Addition over Multiplication];

Axioms of "Real" Numbers **Exponentiation**:

RE0. For all r in $\mathbf{R} - \{\mathbf{0}\}$, $r^{\mathbf{0}} = \mathbf{1}$; the value of the expression $\mathbf{0}^{\mathbf{0}}$ is undefined [Exponentiation Predominance];

RE1. For all $r_1, r_2, r_3 \geq \mathbf{1}$ in \mathbf{R} , $r_1^{r_2} \times r_1^{r_3} = r_1^{r_2 + r_3}$ [Exponentiation Additivity] [$\in \mathbf{AQ}$]; some values $\notin \mathbf{R}$;

RE2. For all $r_1, r_2, r_3 \geq \mathbf{1}$ in \mathbf{R} , $(r_1^{r_2})^{r_3} = r_1^{r_2 \times r_3}$ [Exponentiation Multiplicativity] [$\in \mathbf{AQ}$]; some values $\notin \mathbf{R}$;

RE3. For all $r_1, r_2, r_3 \geq \mathbf{1}$ in \mathbf{R} , $(r_1 \times r_2)^{r_3} = r_1^{r_3} \times r_2^{r_3}$ [Exponentiation "Distributivity"] [$\in \mathbf{AQ}$]; some values $\notin \mathbf{R}$;

RE4. For all r_1, r_2 in \mathbf{R} , $r_1^{r_2}$ is in \mathbf{R} [Exponentiation Closure] [$\in \mathbf{AQ}$]; $\mathbf{0}^{\mathbf{0}} \notin \mathbf{R}$; $\exists r_1 < \mathbf{0} \in \mathbf{R} [r_1^{r_2} \notin \mathbf{R} \text{ if } r_2 \in (\mathbf{0}, \mathbf{1})]$, etc.;

RE5. For all r in \mathbf{R} , $r^{\mathbf{1}} = r$ [Exponentiation Invariance];

RE6. For all $r_1, r_2, r_3 \geq \mathbf{1}$ in \mathbf{R} , $(r_2/r_1)^{r_3} = (r_2/r_1)^{r_3} = r_1^{r_3}/r_2^{r_3}$; $(r_3^{r_2}/r_3^{r_1}) = r_3^{r_2 - r_1}$ [Exponentiation "Real"-ity] [$\in \mathbf{AQ}$];

Axioms of "Real" Numbers **Order**:

RO1. For all r_1, r_2 in \mathbf{R} , either $r_1 > r_2$, or $r_1 = r_2$, or $r_1 < r_2$ [Trichotomy "Law" [First Order]];

RO2. For all $r_1 > \mathbf{0}, r_2 > \mathbf{0}$ in \mathbf{R} , $r_1 + r_2 > r_1, r_2$ [Additive Order for "Positive" Reals [First Order]];

RO3. For all $r_1 > \mathbf{1}, r_2 > \mathbf{1}$ in \mathbf{R} , $r_1 \times r_2 > r_1, r_2$ [Multiplicative Order for "Positive" Reals [First Order]];

$$[\forall r_1, r_2 \in \mathbf{R}] [([r_1 > \mathbf{1} \wedge r_2 > \mathbf{1}] \Rightarrow [r_1 \times r_2 > r_1, r_2]) \wedge ([r_1 \times r_2 = \mathbf{0}] \Leftrightarrow ([r_1 = \mathbf{0}] \vee [r_2 = \mathbf{0}]))];$$

RO4. Any non-empty-set-of "Real" Numbers contains a **least** element [Well Ordering Principle [Second Order]];

$$[\forall S \subset \mathbf{R}] [S \neq \emptyset] [\exists s \in S] [\forall r \in S] [r \geq s];$$

RO5. $\mathbf{0} < \mathbf{1}$ [Order of Invariance Elements [First Order]];

Second Order Axiom of "Real" Numbers "Rationality":

R1. For every r in \mathbf{R} , there is a r_2 in the \mathbf{R} -subset of \mathbf{Z} -like "Reals", $\mathbf{R}_2 \subset \mathbf{R}$, such that $r_2 \times r \in \mathbf{R}_2$;

$$[\exists \mathbf{R}_2 \subset \mathbf{R}] [\forall r \in \mathbf{R}] [\exists r_2 \in \mathbf{R}_2] [r_2 \times r \in \mathbf{R}_2];$$

Second Order Axiom of the "Continuousness" of the "Real" Numbers [Completeness]:

R1. Every Set $S \neq \emptyset$ of "Real" Numbers that has an **upper bound** also has a **least upper bound** [= a "supremum"];

$$[\forall S] [S \subset \mathbf{R}] [S \neq \emptyset] [\exists b \in \mathbf{R}] [\forall s \in S] [\forall b \in \mathbf{B}] [s < b] \Rightarrow [\exists b_{\min} \in \mathbf{B}] [\forall b \in \mathbf{B}] [b_{\min} \leq b] [\in \mathbf{AQ}];$$

C.2.6. stage/step S = 6: Consolidation and ‘self-aporization’ of the System of Arithmetic of the ‘‘Complex Numbers’’, **C**.

6.1. descriptive name of stage: *Consolidation of the ‘‘Complexes’’ and their ‘‘Formal Subsumption’’ by the ‘‘Hamilton numbers’’*.

6.2. stage ‘parametrics’: [total terms #, 2^6] 64; [new terms #, 2^{6-1}] 32; [new terms needing solution #, $(2^{6-1} - 2)$] 30.

6.3. «*aporia*» of this stage: $\frac{3}{H} \underline{C} \dashv \oplus \frac{3}{H} \underline{h}$.

6.4. ‘‘incompleteness’’-revealing ‘‘unsolvable’’ equation of this stage: $\mathbf{x}_6 \mathbf{y}_6 + \mathbf{y}_6 \mathbf{x}_6 = \pm 0 = (\pm 0)\mathbf{r} \pm (\pm 0)\mathbf{j} \pm (\pm 0)\mathbf{j} \pm (\pm 0)\mathbf{k}$.

6.5. ‘‘unsolvable’’ equations’ paradox: *The Paradox of Sign-Reversing Factor-Reversal Multiplication within* $\frac{3}{H} \underline{i}$.

6.6. shortcut rendition of the ‘product-tion’ of stage 6 from stage 5 [via the ‘meta-meristematic principle’]:

$$\begin{aligned} \frac{3}{H} \underline{H}_6^{\#} &= \frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{H}_5^{\#} = \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{N}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{a}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{aN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{m}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{mN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{ma}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{maN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{f}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{fN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{fa}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{faN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{fm}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{fmN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{fma}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{fmaN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{d}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{da}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{daN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dm}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dmN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dma}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dmaN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{df}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfa}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfaN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfm}_5^{\#} \right) \oplus \\ &= \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfmN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfma}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{dfmaN}_5^{\#} \right) \oplus \left(\frac{3}{H} \underline{i}_5^{\#} \otimes \frac{3}{H} \underline{i}_5^{\#} \right) = \\ &= \frac{3}{H} \underline{C} \dashv \oplus \frac{3}{H} \underline{h}. \end{aligned}$$

6.7. ‘pre-solved’ new terms -- new terms of *given* or *already solved* meanings/definitions:

6.7.1. $\equiv \left(\frac{3}{H} \underline{C} \right) \dashv \oplus \frac{3}{H} \underline{C}$ -- first-and/or-higher-order-logic expression of the axioms-system-component for the ‘sub-arithmetic’

of the ‘‘epitome’’ of the ‘‘Complexes’’, $\mathbf{C} \equiv \{ ((+0...01.0...) \mathbf{r} + (\pm 0...0.0...) \mathbf{j}), ((+0...02.0...) \mathbf{r} + (\pm 0...0.0...) \mathbf{j}), \dots \}$,

the sub-space of the **C** number-space which expresses the fruition of the *critique* of $\frac{3}{H} \underline{dfmaN}$ by $\frac{3}{H} \underline{i}$ in the form of the product

$$\frac{3}{H} \underline{i} \otimes \frac{3}{H} \underline{dfmaN} = \frac{3}{H} \underline{i} \otimes \frac{3}{H} \underline{r}, \text{ namely } \frac{3}{H} \underline{idfmaN} = \frac{3}{H} \underline{C}, \text{ the sub-space of the } \mathbf{C} \text{ number-space containing all}$$

$\frac{3}{H} \underline{i}$ -converted $\frac{3}{H} \underline{d}$ -converted $\frac{3}{H} \underline{f}$ -converted $\frac{3}{H} \underline{m}$ -converted $\frac{3}{H} \underline{a}$ -converted ‘‘Natural’’ Numbers, $\mathbf{N} \equiv \{ \mathbf{I}, \mathbf{II}, \mathbf{III}, \dots \}$.

6.7.2. $\equiv \left(\frac{3}{H} \underline{h} \right) \dashv \oplus \frac{3}{H} \underline{h}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘*anti-commuting*’ ‘‘qualitative units’’ [cf. Kline], i.e., of the ‘‘hamiltonian numbers’’, $\mathbf{h} \equiv \{ \mathbf{d}\mathbf{j} + \mathbf{d}\mathbf{k}; \mathbf{i}^2 \frac{1}{2} \mathbf{i},$

& $\mathbf{k}^2 \frac{1}{2} \mathbf{k}$ & $\mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j}$ & $\mathbf{j}\mathbf{j} = \mathbf{k}$; fruition of the [immanent] *self-critique* of $\frac{3}{H} \underline{i}$, namely $\frac{3}{H} \underline{h} \equiv \frac{3}{H} \underline{h}$.

6.8. stage-specific solution, term-by-term [using ‘Organonic Algebraic Method’, Procedure Module 3.]:

6.8.1. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{in} \end{smallmatrix} \right) \right) \mathcal{E} \rightarrow \mathcal{Q}_{33}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**Natural**’ Numbers, $\text{in} \equiv \{ \text{Ir}, \text{Irr}, \text{IIrr}, \dots \}$, the sub-space of the **C** number-space which expresses the fruition of the critique of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{N}$ by $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ in the product $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i} \otimes \begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{in} \end{smallmatrix} \right)$, namely $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{in} \end{smallmatrix} \right)$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{N}$ with/with/by/by/to/and $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ as its uplift into $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{C}$.

6.8.2. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{ia} \end{smallmatrix} \right) \right) \mathcal{E} \rightarrow \mathcal{Q}_{34}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**aught**’ Numbers, $\text{ia} \equiv \{ ((\text{I} - \text{I})\text{r} + (\text{I} - \text{I})\text{j}), ((\text{II} - \text{II})\text{r} + (\text{II} - \text{II})\text{j}), \dots \}$, the sub-space of **C** number-space which expresses fruition for the critique of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{a}$ by $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ in the product $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i} \otimes \begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{a} \end{smallmatrix} \right)$, namely $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{ia} \end{smallmatrix} \right)$, also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{a}$ with/with/by/by/to/and $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ as its uplift into $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{C}$.

6.8.3. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{iaN} \end{smallmatrix} \right) \right) \mathcal{E} \rightarrow \mathcal{Q}_{35}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**aught-ized Natural**’ Numbers, $\text{iaN} \equiv \{ ((0\dots01)\text{r} + (0\dots0)\text{j}), ((0\dots02)\text{r} + (0\dots0)\text{j}), \dots \}$, the sub-space of **C** number-space which expresses the fruition of the critique of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{aN} \end{smallmatrix} \right)$ by $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ in the product $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i} \otimes \begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{aN} \end{smallmatrix} \right)$, namely $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{iaN} \end{smallmatrix} \right)$, “‘hybridization/synthesis/subsumption...’” of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{aN} \end{smallmatrix} \right)$ with/with/by... $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ as uplift of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{aN} \end{smallmatrix} \right)$ up into $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{C}$.

6.8.4. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{im} \end{smallmatrix} \right) \right) \mathcal{E} \rightarrow \mathcal{Q}_{36}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**minus**’ Numbers, $\text{im} \equiv \{ ((0 - 1)\text{r} + (\pm 0)\text{j}), ((0 - 2)\text{r} + (\pm 0)\text{j}), \dots \}$, the sub-space of the **C** number-space which expresses the fruition of the critique of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{m}$ by $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ in the product $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i} \otimes \begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{m} \end{smallmatrix} \right)$, namely $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{im} \end{smallmatrix} \right)$.

6.8.5. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{imN} \end{smallmatrix} \right) \right) \mathcal{E} \rightarrow \mathcal{Q}_{37}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**sign-ized Natural**’ Numbers, $\text{imN} \equiv \{ +1\text{r}, +2\text{r}, +3\text{r}, \dots \}$, the sub-space of **C** number-space which expresses the fruition of the critique of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{mN} \end{smallmatrix} \right)$ by $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ in the product $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i} \otimes \begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{mN} \end{smallmatrix} \right)$, namely $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{imN} \end{smallmatrix} \right)$.

6.8.6. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{ima} \end{smallmatrix} \right) \right) \mathcal{E} \rightarrow \mathcal{Q}_{38}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the ‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**sign-ized aught**’ Numbers, $\text{ima} \equiv \{ (\pm(\text{I} - \text{I})\text{r} + (\pm(\text{I} - \text{I})\text{j})), \dots \}$, the sub-space of the **C** number-space which expresses the fruition of the critique of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{ma} \end{smallmatrix} \right)$ by $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ in the product $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i} \otimes \begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{ma} \end{smallmatrix} \right)$, namely $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{ima} \end{smallmatrix} \right)$, which is also the “‘hybridization/synthesis/subsumption/conversion...’” of $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \left(\begin{smallmatrix} \# \\ \text{ma} \end{smallmatrix} \right)$ with/with/by/by... $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{i}$ as its elevation up into $\begin{smallmatrix} 3 \\ H \end{smallmatrix} \text{C}$.

6.8.7. $\equiv \left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \text{imaN} \# \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right)_{39}$

‘sub-arithmetic’ of the ‘**Complex**-ized’ ‘*sign*-ized’ ‘aught-ized’ “**N**atural Numbers”, **imaN** $\equiv \{ (+0...01)r + (\pm 0...0)i, \dots \}$, the

sub-space of **C** number-space which expresses the fruition of the critique of $\text{H}^{\#}_{\text{Q}} \text{maN}$ by $\text{H}^{\#}_{\text{I}}$ in the product $\text{H}^{\#}_{\text{I}} \otimes \text{H}^{\#}_{\text{Q}} \text{maN}$,
namely $\text{H}^{\#}_{\text{Q}} \text{imaN}$, “‘hybridization/synthesis/subsumption...’” of $\text{H}^{\#}_{\text{a}}$ with/with/by... $\text{H}^{\#}_{\text{I}}$ as elevation of $\text{H}^{\#}_{\text{Q}} \text{maN}$ up into $\text{H}^{\#}_{\text{C}}$.

6.8.8. $\equiv \left(\begin{matrix} 3 \\ H \end{matrix} \right)_{\text{if}}^{\#} \mapsto \boxed{40}$

‘sub-arithmetic’ of the ‘**Complex-ized**’ **fractional** Numbers’, **if** $\equiv \{ (\mathbf{fr} + (0/\mathbf{z}_2)\mathbf{j}) = ((\mathbf{z}_1/\mathbf{z}_2)\mathbf{r} + (0/\mathbf{z}_2)\mathbf{j}) \mid 0 \neq \mathbf{z}_2, \& \mathbf{z}_1, \mathbf{z}_2 \in \mathbf{m} \}$,

the sub-space of **C** number-space which expresses fruition for the critique of $\overset{3}{\underset{\#}{\text{H}}}\overset{\text{f}}{\text{f}}$ by $\overset{3}{\underset{\#}{\text{H}}}\overset{\text{i}}{\text{i}}$ in the product $\overset{3}{\underset{\#}{\text{H}}}\overset{\text{i}}{\text{i}} \otimes \overset{3}{\underset{\#}{\text{H}}}\overset{\text{f}}{\text{f}}$, namely

3_H[#]_{if}, the ““hybridization/synthesis/subsumption/conversion...”” of 3_H[#]_a with/by/by... 3_H[#]_i as the elevation of 3_H[#]_f into 3_H[#]_c.

6.8.9. $\equiv \left(\begin{matrix} 3 \\ H \\ \text{if } N \end{matrix} \right) \mapsto \boxed{}_{41}$

'sub-arithmetic' of the 'Complex-ized' 'fraction-ized' "Natural Numbers", $\text{if } \mathbf{N} \equiv \{ (1/1)\mathbf{r}, (2/1)\mathbf{r}, (3/1)\mathbf{r}, \dots \}$, the sub-space of the **C**

number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{fN}}$ by $\overset{3}{\text{H}}_{\text{I}}^{\#}$ in the product $\overset{3}{\text{H}}_{\text{I}}^{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{fN}}$, namely $\overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{fN}}$,

““hybridization/synthesis/subsumption/conversion/adjustment...”” of $\overset{3}{\text{H}}\overset{\text{a}}{\text{H}}_{\#}$ with/with/by/by/to... $\overset{3}{\text{H}}\overset{\text{i}}{\text{H}}_{\#}$ as the elevation of $\overset{3}{\text{H}}\overset{\text{g}}{\text{H}}_{\#}$ to $\overset{3}{\text{H}}\overset{\text{c}}{\text{H}}_{\#}$.

6.8.10. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{ifa} \end{smallmatrix} \right) \right) \mapsto \begin{smallmatrix} \boxed{} \\ \boxed{} \end{smallmatrix}_{42}$

'sub-arithmetic' of the '**Complex-ized**' '**fraction-ized**' '**ought** Numbers', **ifa** $\equiv \{ ((I - I)/I)r + ((I - I)/I)i, \dots \}$, the sub-space of the **C**

number-space which expresses fruition for the critique of $\overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\text{fa}}{\text{f}}$ by $\overset{3}{\text{H}}\overset{\text{f}}{\text{H}}\overset{\#}{\text{f}}$ in the product $\overset{3}{\text{H}}\overset{\text{f}}{\text{H}}\overset{\#}{\text{f}} \otimes \overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\text{fa}}{\text{f}}$, namely $\overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\text{fa}}{\text{f}}$, also the


“‘hybridization/synthesis/subsumption/conversion/adjustment...’” of $\overset{3}{\text{H}}\overset{\text{a}}{\text{a}}_{\#}$ with/with/by/by/to... $\overset{3}{\text{H}}\overset{\text{i}}{\text{i}}_{\#}$ as $\overset{3}{\text{H}}\overset{\text{a}}{\text{a}}_{\#}$ s uplift into $\overset{3}{\text{H}}\overset{\text{c}}{\text{c}}_{\#}$.

6.8.11. $\equiv \left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{ifan} \end{smallmatrix} \right) \mapsto \left(\begin{smallmatrix} \text{ } \\ \text{ } \end{smallmatrix} \right)_{43}$

‘sub-arithmetic’ of the ‘**Complex**-ized’ ‘**fraction**-ized’ ‘**aught**-ized’ “**N**aturals”, **ifaN** $\equiv \{ (0...01/0...01)r + (0...0/0...01)i, \dots \}$, the

sub-space of the **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\underset{\text{faN}}{\text{g}}}$ by $\overset{3}{\text{H}}\overset{\text{I}}{\underset{\#}{\text{I}}}$ in the product $\overset{3}{\text{H}}\overset{\text{I}}{\underset{\#}{\text{I}}} \otimes \overset{3}{\text{H}}\overset{\#}{\underset{\text{faN}}{\text{g}}}$,

namely $\overset{3}{\text{H}}\overset{\#}{\text{ifaN}}$, which is also the ““hybridization/synthesis/subsumption/conversion...”” of $\overset{3}{\text{H}}\overset{\#}{\text{ifaN}}$ with/with/by/by... $\overset{3}{\text{H}}\overset{\#}{\text{ifaN}}$ as

3[#]_H  s elevation from down in 3^R_H to up into 3^C_H.

6.8.12. $\equiv \left(\left(\overset{3}{H} \right) \left(\overset{\#}{\text{ifm}} \right) \right) \Rightarrow \left(\right)_{44}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

'sub-arithmetic' of the 'Complex-ized' 'fraction-ized' 'minus numbers', $\mathbf{ifm} \equiv \{ ((0-1)/+1)r + (\pm(0-0)/+1)i, \dots \}$, the

sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{fm}}$ by $\overset{3}{\text{H}}\overset{\text{i}}{\text{I}}_{\#}$ in the product $\overset{3}{\text{H}}\overset{\text{i}}{\text{I}}_{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{fm}}$,

namely $\overset{3}{\text{H}}\overset{\#}{\underset{\text{ifm}}{\text{g}}}$, also the ““hybridization/synthesis/subsumption/conversion”” of $\overset{3}{\text{H}}\overset{\#}{\underset{\text{fm}}{\text{g}}}$ with/with/by/by $\overset{3}{\text{H}}\overset{\#}{\text{i}}$ up into $\overset{3}{\text{H}}\overset{\#}{\text{c}}$.

6.8.13. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\text{ifmN} \right) \right) \Rightarrow \left(\text{45} \right)$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘Complex-ized’ ‘fraction-ized’ ‘sign-ized’ “Natural Numbers”, $\text{ifmN} \equiv \{ (+\text{I}/+\text{I})r, (+\text{II}/+\text{II})r, (+\text{III}/+\text{III})r, \dots \}$,





the sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\circ}{\underset{\text{f m N}}{\text{O}}}$ $\#$ by $\overset{3}{\text{H}}\overset{\text{I}}{\underset{\#}{\text{I}}}$ in the product $\overset{3}{\text{H}}\overset{\text{I}}{\underset{\#}{\text{I}}} \otimes \overset{3}{\text{H}}\overset{\circ}{\underset{\text{f m N}}{\text{O}}}$ $\#$,

namely $\overset{3}{\text{H}}\overset{\#}{\underset{\text{ifmN}}{\text{Q}}}$, also the ““hybridization/synthesis/subsumption/conversion”” of $\overset{3}{\text{H}}\overset{\#}{\underset{\text{fmN}}{\text{Q}}}$ with/with/by/by $\overset{3}{\text{H}}\overset{\#}{\text{I}}$ up into $\overset{3}{\text{H}}\overset{\#}{\text{C}}$.

6.8.14. $\equiv \left(\begin{matrix} 3 \\ H \end{matrix} \right) \left(\begin{matrix} \# \\ ifma \end{matrix} \right) E \Rightarrow \boxed{46}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the 'Complex-ized' 'fraction-ized' 'sign-ized ought Numbers', **ifma** $\equiv \{ (\pm 0/+1)r + (\pm 0/+1)i, \dots \}$, the sub-space

of the **C** number-space which expresses the fruition of the critique of $\overset{3}{\underset{\text{fma}}{\text{H}}}\overset{\#}{\text{G}}$ by $\overset{3}{\underset{\#}{\text{H}}}\overset{\text{i}}{\text{I}}$ in the product $\overset{3}{\underset{\#}{\text{H}}}\overset{\text{i}}{\text{I}} \otimes \overset{3}{\underset{\text{fma}}{\text{H}}}\overset{\#}{\text{G}}$, namely

3  #, also the ““hybridization/synthesis/subsumption/conversion...”” of 3  # with/with/by/by... 3  # up into 3  #.

6.8.15. $\equiv \langle \langle \overset{3}{H} \rangle \rangle_{\text{ifmaN}}^{\#} \models \sqcup_{47}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘**C**omplex-ized’ ‘**f**raction-ized’ ‘**s**ign-ized **a**ught-ized “**N**aturals”, **ifmaN** $\equiv \{ (+0...01/+0...01)r + (\pm 0...0/+0...01)i, \dots \}$, the

sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\underset{\text{fmaN}}{\text{g}}}$ by $\overset{3}{\text{H}}\overset{\#}{\text{i}}$ in the product $\overset{3}{\text{H}}\overset{\#}{\text{i}} \otimes \overset{3}{\text{H}}\overset{\#}{\underset{\text{fmaN}}{\text{g}}}$,

namely $\overset{3}{\text{H}}\overset{\#}{\text{g}}\text{ifmaN}$, which is also the “hybridization/synthesis/subsumption/conversion/adjustment/reconciliation” of $\overset{3}{\text{H}}\overset{\#}{\text{g}}\text{fmaN}$

with/with/by/by/to/and $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{I} \\ \# \end{matrix}$ as the elevation of $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{G} \\ \# \end{matrix}$ from down in $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{R} \\ \# \end{matrix}$ to up into $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{C} \\ \# \end{matrix}$

6.8.16. $\equiv \left(\left(\overset{3}{H} \right) \left(\overset{\#}{Q} \right) \right) \vdash \Box_{48}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘**Complex-ized**’ ‘**d**ecimal Numbers’, $\mathbf{id} \equiv \{\mathbf{dr} + (\mathbf{0.0...})\mathbf{j} \mid \forall \mathbf{d} \in \mathbf{d}\}$, the sub-space of the **C** number-space which

expresses the fruition of the critique of $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \begin{smallmatrix} \text{G} \\ \text{d} \end{smallmatrix} \begin{smallmatrix} \# \\ \text{d} \end{smallmatrix}$ by $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \begin{smallmatrix} \text{i} \\ \# \end{smallmatrix}$ in the product $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \begin{smallmatrix} \text{i} \\ \# \end{smallmatrix} \otimes \begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \begin{smallmatrix} \text{G} \\ \text{d} \end{smallmatrix} \begin{smallmatrix} \# \\ \text{d} \end{smallmatrix}$, namely $\begin{smallmatrix} 3 \\ \text{H} \end{smallmatrix} \begin{smallmatrix} \text{G} \\ \text{id} \end{smallmatrix} \begin{smallmatrix} \# \\ \text{d} \end{smallmatrix}$, which is also the

“hybridization/synthesis/subsumption/conversion/adjustment/reconciliation” of $\mathbf{3}_{\mathbf{H}^{\#}}^{\mathbf{d}}$ with/by/to/and $\mathbf{3}_{\mathbf{H}^{\#}}^{\mathbf{i}}$ as the elevation of

from down in $\text{H}-\text{R}_{\#}$ to up into $\text{H}-\text{C}_{\#}$.

6.8.17. $\equiv \left(\left(\overset{3}{H} \right) \left(\overset{\#}{idN} \right) \right) \vdash \vdash_{49}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

C**omplex**-ized' '**d****e****c****i****m****a****l**-ized' "**N**atural" Numbers, **idN** $\equiv \{ ((\text{I})\text{---})_r, ((\text{II})\text{---})_r, ((\text{III})\text{---})_r, \dots \}$, the sub-space of the **C**

number-space which expresses the fruition of the *critique* of $\overset{3}{\text{H}}\overset{\#}{\text{G}}\text{dN}$ by $\overset{3}{\text{H}}\text{i}_{\#}$ in the product $\overset{3}{\text{H}}\text{i}_{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{G}}\text{dN}$, namely $\overset{3}{\text{H}}\overset{\#}{\text{G}}\text{idN}$.

which is also the ““hybridization/synthesis/subsumption/conversion/adjustment...”” of $\overset{3}{\text{H}}\overset{\#}{\underset{\text{dN}}{\text{C}}}$ with/with/by/by/to... $\overset{3}{\text{H}}\overset{\#}{\text{C}}_{\#}$ up into $\overset{3}{\text{H}}\overset{\#}{\text{C}}_{\#}$.

6.8.18. $\equiv \left(\left(\overset{3}{H} \right) \left(\overset{\#}{\text{ida}} \right) \right) \mapsto \boxed{50}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the 'Complex-ized' 'decimal-ized' 'ought' Numbers, $\text{ida} \equiv \{ ((\text{I}-\text{I}).0...)r + ((\text{I}-\text{I}).0...)i, \dots \}$, the sub-space of

the **C** number-space which expresses the fruition of the critique of $\overset{3}{\underset{\text{H}}{\text{G}}}_{\text{da}}^{\#}$ by $\overset{3}{\underset{\text{H}}{\text{I}}}_{\#}$ in the product $\overset{3}{\underset{\text{H}}{\text{I}}}_{\#} \otimes \overset{3}{\underset{\text{H}}{\text{G}}}_{\text{da}}^{\#}$, namely

$\overset{3}{\text{H}}\overset{\#}{\text{Q}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}$

with/with/by/by/to/and $\overset{3}{\text{H}}\underset{\#}{\text{I}}$ as the elevation of $\overset{3}{\text{H}}\underset{\#}{\text{Q}}$ from down in $\overset{3}{\text{H}}\underset{\#}{\text{R}}$ to up into $\overset{3}{\text{H}}\underset{\#}{\text{C}}$.

6.8.19. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{idaN}} \right) \right) \Rightarrow \boxed{51}$ -- the first-and/or-higher-order-logic expression of the axioms-system-component for the

'sub-arithmetic' of the '**Complex-ized**' '**decimal-ized**' '**ought-ized** **Natural Numbers**', $\text{id}\mathbf{a}\mathbf{N} \equiv \{ (0\ldots 01.0\ldots)\mathbf{r} + (0\ldots 0.0\ldots)\mathbf{i}, \ldots \}$,

the sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{daN}$ by $\overset{3}{\text{H}}\overset{\text{I}}{\text{\#}}$ in the product $\overset{3}{\text{H}}\overset{\text{I}}{\text{\#}} \otimes \overset{3}{\text{H}}\overset{\text{Q}}{\text{daN}}$,

namely ${}^3\text{H}\text{Q}^{\#}\text{idaN}$, which is also the ““hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”” of ${}^3\text{H}\text{Q}^{\#}\text{daN}$

with/with/by/by/to/and $\text{H}^{\text{3}\text{I}}_{\text{#}}$ the elevation of $\text{H}^{\text{3}\text{Q}}_{\text{#}}$ from down in $\text{H}^{\text{3}\text{R}}_{\text{#}}$ to up into $\text{H}^{\text{3}\text{C}}_{\text{#}}$

6.8.20. $\equiv \left(\left(\overset{3}{\text{H}} \right) \left(\overset{\#}{\text{idm}} \right) \right) \vdash \vdash \vdash$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the '**Complex-ized**' '**decimal-ized**' '**m**inus' Numbers, $\text{idm} \equiv \{ ((0-1).0...)r + (\pm(0-0).0...)i, \dots \}$, the sub-space

of the **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{g}}\overset{\text{dm}}{\text{dm}}$ by $\overset{3}{\text{H}}\overset{\text{dm}}{\text{dm}}$ in the product $\overset{3}{\text{H}}\overset{\text{dm}}{\text{dm}} \otimes \overset{3}{\text{H}}\overset{\#}{\text{g}}\overset{\text{dm}}{\text{dm}}$, namely

$3H \overset{\#}{\underset{\text{idm}}{\text{Q}}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $3H \overset{\#}{\underset{\text{dm}}{\text{Q}}}$

with/with/by/by/to/and $\text{H}^{\text{3}}_{\text{H}^{\#}}$ as the elevation of $\text{H}^{\text{3}}_{\text{H}^{\#}}$ from down in $\text{H}^{\text{3}}_{\text{H}^{\#}}$ to up into $\text{H}^{\text{3}}_{\text{H}^{\#}}$.

6.8.21. $\equiv \left(\begin{array}{c} 3 \\ H \end{array} \right) \left(\begin{array}{c} \# \\ idmN \end{array} \right) E \rightarrow \boxed{53}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of the ‘Complex-ized’ ‘decimal-ized’ ‘*sign*-ized Natural Numbers’, **idmN** $\equiv \{ (+1....)r + (\pm 0....)i, ... \}$, the

sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}} \overset{\#}{\text{G}}_{\text{dmN}}$ by $\overset{3}{\text{H}}_{\#}$ in the product $\overset{3}{\text{H}}_{\#} \otimes \overset{3}{\text{H}} \overset{\#}{\text{G}}_{\text{dmN}}$,

namely $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{idmN}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{dmN}$

with/by/to/and $\begin{matrix} 3 \\ \text{H}^\sharp \end{matrix}$ as the elevation of $\begin{matrix} 3 \\ \text{H}^\flat \end{matrix}$ from down in $\begin{matrix} 3 \\ \text{H}^\flat \end{matrix}$ to up into $\begin{matrix} 3 \\ \text{H}^\sharp \end{matrix}$

6.8.22. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \text{idma} \end{smallmatrix} \right) \right) \mapsto \left(\begin{smallmatrix} 54 \end{smallmatrix} \right)$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

'sub-arithmetic' of the 'Complex-ized' 'decimal-ized' 'sign-ized ought Numbers', $\text{idma} \equiv \{ (\pm (\text{I}-\text{I}).0\dots)\text{r} + (\pm (\text{I}-\text{I}).0\dots)\text{i}, \dots \}$,

sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{G}}\overset{\text{dma}}{\text{}}$ by $\overset{3}{\text{H}}\overset{\text{I}}{\#}$ in the product $\overset{3}{\text{H}}\overset{\text{I}}{\#} \otimes \overset{3}{\text{H}}\overset{\text{G}}{\text{dma}}$,





namely $\overset{3}{\text{H}}\overset{\#}{\text{G}}\text{idma}$, which is also the ““hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”” of $\overset{3}{\text{H}}\overset{\#}{\text{G}}\text{idma}$

with/with/by/by/to/and $\overset{3}{\text{H}}\text{--}\overset{\#}{\text{I}}$ as the elevation of $\overset{3}{\text{H}}\text{--}\overset{\#}{\text{I}}$ from down in $\overset{3}{\text{H}}\text{--}\overset{\#}{\text{R}}$ to up into $\overset{3}{\text{H}}\text{--}\overset{\#}{\text{C}}$.

6.8.23. $\equiv \left(\begin{matrix} 3 \\ H \end{matrix} \right) \text{idmaN}^{\#} \vdash \vdash_{55}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘Complex-ized’ ‘decimal-ized’ ‘sign-ized’ ‘aught-ized’ “Natural Numbers”, $\text{idmaN} \equiv \{ (+0...01.0...)r + (\pm 0...0.0...)i, \dots \}$, the

sub-space of **C** number-space which expresses fruition for the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\overset{\#}{\text{dman}}\overset{\#}{\text{N}}$ by $\overset{3}{\text{H}}\overset{\#}{\text{I}}\overset{\#}{\text{N}}$ in the product $\overset{3}{\text{H}}\overset{\#}{\text{I}}\overset{\#}{\text{N}}\otimes\overset{3}{\text{H}}\overset{\#}{\text{Q}}\overset{\#}{\text{dman}}\overset{\#}{\text{N}}$,

namely  , also the ““hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”” of  

with/with/by/by/to/and $\overset{3}{\underset{\#}{H}}_{\text{I}}$ as the elevation of $\overset{3}{\underset{\#}{H}}_{\text{dman}}$ from down in $\overset{3}{\underset{\#}{H}}_{\text{R}}$ to up into $\overset{3}{\underset{\#}{H}}_{\text{C}}$.

6.8.24. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{idf} \end{smallmatrix} \right) \right) \vdash \vdash_{56}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘sub-arithmetic’ of ‘Complex-ized’ ‘decimal-ized’ ‘fractional Numbers’, $\text{idf} \equiv \{ ((z_1/z_2)...)r + ((\pm 0/z_2)...)i \mid 0 \neq z_2, z_1 \in \mathbf{m} \}$,

the sub-space of the **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{df}}$ by $\overset{3}{\text{H}}\overset{\#}{\text{I}}_{\#}$ in the product $\overset{3}{\text{H}}\overset{\#}{\text{I}}_{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{df}}$,

namely $\overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{idf}}$, which is also the ““hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”” of $\overset{3}{\text{H}}\overset{\#}{\text{G}}_{\text{df}}$

with/with/by/by/to/and $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{I} \\ \# \end{matrix}$ as the elevation of $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{G} \\ \text{df} \end{matrix} \begin{matrix} \# \\ \end{matrix}$ from down in $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{R} \\ \# \end{matrix}$ to up into $\begin{matrix} 3 \\ \text{H} \end{matrix} \begin{matrix} \text{C} \\ \# \end{matrix}$.

6.8.25. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{idfN} \end{smallmatrix} \right) \right) \vdash \left(\begin{smallmatrix} \boxed{} \\ 57 \end{smallmatrix} \right)$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

the **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\underset{\text{dfn}}{\text{G}}}$ by $\overset{3}{\text{H}}\overset{\#}{\text{I}}$ in the product $\overset{3}{\text{H}}\overset{\#}{\text{I}} \otimes \overset{3}{\text{H}}\overset{\#}{\underset{\text{dfn}}{\text{G}}}$, namely

by/by/to/and $\text{H}_{\#}^{\text{3}} \text{dfN}$ as the elevation of $\text{H}_{\#}^{\text{3}} \text{dfN}$ from down in $\text{H}_{\#}^{\text{3}} \text{R}$ to up into $\text{H}_{\#}^{\text{3}} \text{C}$.

6.8.26. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{idfa} \end{smallmatrix} \right) \right) \mapsto \boxed{58}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

the sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}} \textcircled{\text{d} \text{f} \text{a}}^{\#}$ by $\overset{3}{\text{H}} \textcircled{\text{i}}_{\#}$ in the product $\overset{3}{\text{H}} \textcircled{\text{i}}_{\#} \otimes \overset{3}{\text{H}} \textcircled{\text{d} \text{f} \text{a}}^{\#}$

namely $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{idfa}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”’ of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{dfa}$ with/by/to/and $\overset{3}{\text{H}}\text{i}_{\#}$ as the elevation of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}\text{dfa}$ from down in $\overset{3}{\text{H}}\text{R}_{\#}$ to up into $\overset{3}{\text{H}}\text{C}_{\#}$.


6.8.27. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{idfaN} \end{smallmatrix} \right) \right) \mapsto \left(\begin{smallmatrix} \boxed{} \\ 59 \end{smallmatrix} \right)$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the






‘Complex-ized’ ‘decimal-ized’ ‘fraction-ized’ ‘ought-ized’ “Naturals” --

$$\text{idfaN} \equiv \{ ((0\dots 01/0\dots 01).0\dots)r + ((0\dots 0/0\dots 0).0\dots)i, \dots \}$$

-- the sub-space of **C** number-space which expresses fruition for the critique of $\overset{3}{\text{H}}\overset{\#}{\text{g}}$ by $\overset{3}{\text{H}}\overset{\#}{\text{i}}$ in the product --

3 H₂ \otimes 3 H \rightarrow dfan #

-- namely  [#], which is also the ““hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”” of

$\begin{matrix} 3 \\ \text{H} \end{matrix}$  $\begin{matrix} \# \\ \text{dfaN} \end{matrix}$ with/with/by/by/to/and $\begin{matrix} 3 \\ \text{H} \end{matrix}$  as the elevation of $\begin{matrix} 3 \\ \text{H} \end{matrix}$  $\begin{matrix} \# \\ \text{dfaN} \end{matrix}$ from down in $\begin{matrix} 3 \\ \text{H} \end{matrix}$  to up into $\begin{matrix} 3 \\ \text{H} \end{matrix}$  $\begin{matrix} \# \\ \text{C} \end{matrix}$

6.8.28. $\equiv \left(\left(\begin{smallmatrix} 3 \\ H \end{smallmatrix} \right) \left(\begin{smallmatrix} \# \\ \text{idfm} \end{smallmatrix} \right) \right) \mapsto \left(\begin{smallmatrix} \text{ } \\ 60 \end{smallmatrix} \right)$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘Complex-ized’ ‘decimal-ized’ ‘fraction-ized’ ‘minus numbers’, **idfm** $\equiv \{(((0-1)/+1)...)r + ((\pm(0-0)/+1)...)i, \dots\}$, the

sub-space of **C** number-space which expresses the fruition of the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{dfm}}$ by $\overset{3}{\text{H}}_{\text{I}_{\#}}$ in the product $\overset{3}{\text{H}}_{\text{I}_{\#}} \otimes \overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{dfm}}$,

namely $\overset{3}{\text{H}} \overset{\#}{\text{Q}}_{\text{idfm}}$, which is also the ““hybridization/synthesis/subsumption/conversion/adjustment/reconciliation”” of $\overset{3}{\text{H}} \overset{\#}{\text{Q}}_{\text{dfm}}$

with/with/by/by/to/and $\begin{matrix} \text{3} \\ \text{H-}\sharp \end{matrix}$ as the elevation of $\begin{matrix} \text{3} \\ \text{H} \end{matrix}$ from down in $\begin{matrix} \text{3} \\ \text{H-}\sharp \end{matrix}$ to up into $\begin{matrix} \text{3} \\ \text{H-C}\sharp \end{matrix}$

6.8.29. $\equiv \left(\left(\overset{3}{H} \right) \left(\overset{\#}{\text{idfmN}} \right) \right) \vdash \left(\left(\right) \right)_{61}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘**C**omplex-ized’ ‘**d**ecimal-ized’ ‘**f**raction-ized’ ‘**s**ign-ized “**N**atural” Numbers, **idfmN** $\equiv \{ ((+1/+1)....)r + ((\pm 0/+1)....)i, ... \}$, the

sub-space of **C** number-space which expresses fruition for the critique of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{dfmN}}$ by $\overset{3}{\text{H}}_{\text{I}}^{\#}$, in product $\overset{3}{\text{H}}_{\text{I}}^{\#} \otimes \overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{dfmN}}$, namely $\overset{\#}{\text{Q}}_{\text{idfmN}}$, also being the “hybridization/synthesis/subsumption/conversion/adjustment/reconciliation” of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{dfmN}}$ with/with/by/by/to/and $\overset{3}{\text{H}}_{\text{I}}^{\#}$ as the elevation of $\overset{3}{\text{H}}\overset{\#}{\text{Q}}_{\text{dfmN}}$ from down in $\overset{3}{\text{H}}_{\text{R}}^{\#}$ to up into $\overset{3}{\text{H}}_{\text{C}}^{\#}$.

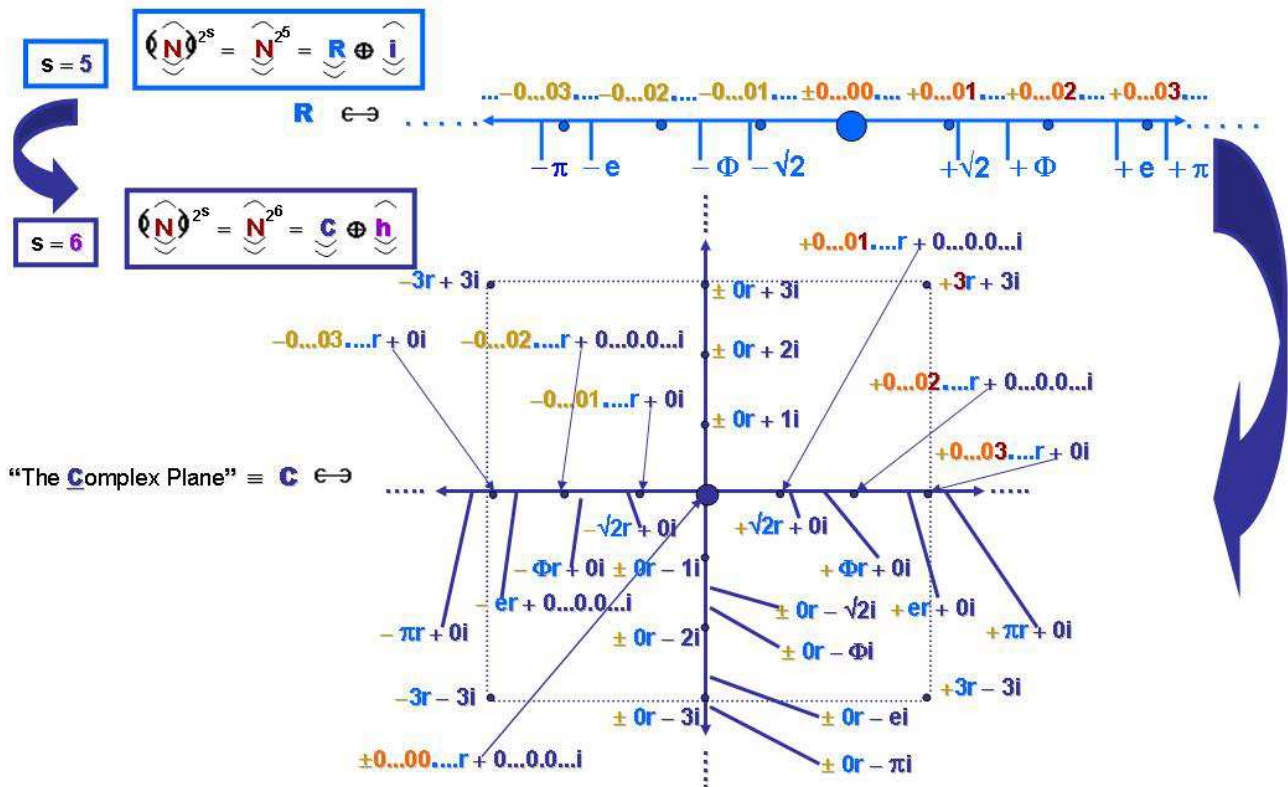
6.8.30. $\equiv \langle \langle \text{H}^3 \rangle_{\text{idfma}}^\# \rangle \vdash \Box_{62}$ -- the first-*and/or-higher-order-logic* expression of the axioms-system-component for the

‘Complex-ized’ ‘decimal-ized’ ‘fraction-ized’ ‘sign-ized’ ‘aught’ numbers, $\text{idfma} \equiv \{ ((\pm(I-I)/+1)...)r + ((\pm(I-I)/+1)...)i, \dots \}$,

the sub-space of **C** number-space which means fruition for the critique of $\overset{\#}{\underset{\text{dfma}}{\text{H}}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$ by $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$ in the product $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}} \otimes \overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$, namely $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$, which is also the “‘hybridization/synthesis/subsumption/conversion/adjustment/reconciliation’” of $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$ with/with/by/by/to/and $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$ as the elevation of $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$ from down in $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$ to up into $\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}\overset{\text{3}}{\text{H}}$.

Number-Line Cumulative Progression Analytic-“Geometric” Model
of the **Gödelian Dialectic of the Standard Arithmetics**:

The Transition from $s = 5$ to $s = 6$, and from $\mathbf{R} \oplus \dots$ to $\mathbf{C} \oplus \dots$



[illegible]

$3_{H\#}^{\text{C}} = 3_{H\#}^{\text{R}} \oplus 3_{H\#}^{\text{I}} \oplus 3_{H\#}^{\text{IN}} \oplus 3_{H\#}^{\text{ia}} \oplus 3_{H\#}^{\text{iaN}} \oplus 3_{H\#}^{\text{im}} \oplus 3_{H\#}^{\text{imN}} \oplus 3_{H\#}^{\text{ima}} \oplus 3_{H\#}^{\text{imaN}}$
 $3_{H\#}^{\text{if}} \oplus 3_{H\#}^{\text{ifN}} \oplus 3_{H\#}^{\text{ifa}} \oplus 3_{H\#}^{\text{ifaN}} \oplus 3_{H\#}^{\text{ifm}} \oplus 3_{H\#}^{\text{ifmN}}$
 $3_{H\#}^{\text{ifma}} \oplus 3_{H\#}^{\text{ifmaN}} \oplus 3_{H\#}^{\text{id}} \oplus 3_{H\#}^{\text{idN}} \oplus 3_{H\#}^{\text{ida}} \oplus 3_{H\#}^{\text{idaN}}$
 $3_{H\#}^{\text{idm}} \oplus 3_{H\#}^{\text{idmN}} \oplus 3_{H\#}^{\text{idma}} \oplus 3_{H\#}^{\text{idmaN}} \oplus 3_{H\#}^{\text{idf}} \oplus 3_{H\#}^{\text{idfN}}$
 $3_{H\#}^{\text{idfa}} \oplus 3_{H\#}^{\text{idfaN}} \oplus 3_{H\#}^{\text{idfm}} \oplus 3_{H\#}^{\text{idfmN}} \oplus 3_{H\#}^{\text{idfma}} \oplus 3_{H\#}^{\text{idfmaN}}$

$$\mathfrak{e}(3_{H\#}) \equiv 3_{H\#}$$

Vignette #4 *The Gödelian Dialectic of the Standard Arithmetics* 4.4.0.II -51 by M. Detonacciones, [Foundation Encyclopedia Dialectica](#) [F.E.D.]

More specifically, is there a kind-of-number(s) that is supplementary-opposite to the $\overset{3}{H}_{\#}^i$ kind-of-number(s); that corresponds to the fruition of a self-critique of $\overset{3}{H}_{\#}^i$, namely, to $\overset{3}{H}_{\#}^i \otimes \overset{3}{H}_{\#}^i = \Delta(\overset{3}{H}_{\#}^i)$? The $s=6$, $\overset{3}{H}_{\#}^C$ stage/step of this dialectic of Standard Number Systems/Standard Arithmetic Systems, is one

dominated by the $\overset{3}{H}_{\#}^i$ «aufheben» operator, through its operation upon / interaction with, and via its “‘synthesis’” with / “‘hybridization’” with / ‘complex unification’ with / “‘real subsumption’” of, every arithmetical axioms-system *component/kind*-of-number(s) *term* that was already evoked in all previous steps/stages, with including $\overset{3}{H}_{\#}^i$ itself.

This includes such interaction with the $\overset{3}{H}_{\#}^N$ «arché»-system, and also with all previous ‘partial uni-thesis’ and ‘full uni-thesis’ ‘axioms-sub-system’ terms, and with all preceding ‘contra-thesis’ number(s)/arithmetical kinds terms, including with $\overset{3}{H}_{\#}^i$ itself as previous-stage ‘meta-meristematic’/“‘vanguard’” ‘contra-thesis’ term -- its own ‘real self-subsumption’/‘self-«aufheben»’/‘self-negation’/‘self-elevation’/‘self-conservation’ -- which yields a new ‘meta-meristematic’/“‘vanguard’” ‘contra-thesis’ ‘axioms-sub-system’ term --

$$\overset{3}{H}_{\#}^i \otimes \overset{3}{H}_{\#}^i = \overset{3}{H}_{\#}^i \oplus \overset{3}{H}_{\#}^i = \overset{3}{H}_{\#}^i \oplus \Delta(\overset{3}{H}_{\#}^i)$$

This ‘i-ization’ of the $\overset{3}{H}_{\#}^i$ -- the “addition”, to the “‘solid’” number “line” of the $s=5$ ‘sub-diagram’ [first rendered in Background section **B.η** of Part **I**.] depicting the $\overset{3}{H}_{\#}^i$, of the potentially-infinite “continuity” of “points” representing the “imaginary” numbers, as a second, qualitatively-different number “line”, perpendicular to the “Real” number “line”, and intersecting the “Real” number “line” at -- having “in common” with it -- only $0 \equiv 0r + 0i$, to form the $s=6$ sub-diagram of that diagram, for the “Complex” ‘number-plane’ -- has worked a further transformation in point-of-view, beyond the viewpoints native to the $\overset{3}{H}_{\#}^N$ and $\overset{3}{H}_{\#}^W$ and $\overset{3}{H}_{\#}^Z$

and $\overset{3}{H}_{\#}^Q$ and $\overset{3}{H}_{\#}^R$ systems before it. It has asserted a view that, not only “Real units”, plus also ‘ratio-nal’ fractions of those units, plus also incommensurable parts -- ‘ir-ratio-nal parts’ -- of such units “count”, & “can count”, but also that so-called “imaginary” units “count”, & “can count”, including that ‘ratio-nal’ fractions of those units, plus also incommensurable parts -- ‘ir-ratio-nal parts’ -- of such units, also “count”. Thus, this “Complex” numbers axiomatic system of arithmetic continues an explicit recognition of the ‘part-iality’ of “‘imaginary finite count-numbers’”, within which, even given their “density”, “‘ratio-nal imaginary numbers’” are but the rarest of exceptions, in terms of the proportion of any extensive finite sub-totality of the “‘ratio-nal imaginary numbers’” to any corresponding finite totality of “imaginary” numbers in general. It so continues BEYOND the limits of a single number-“line”/“number-dimension”, to posit a second dimension of number, a two-dimensional ‘number-plane’, seen as continuously-populated with ‘number-points’, including with radical-derived and/or transcendental-functions-derived ‘number-points’, in every direction within that new ‘number-plane’.

We might suspect that the $s=6$ stage’s also-emergent ‘supplementary self-opposition’ of/to/within the $\overset{3}{H}_{\#}^C$ system -- its ‘supplementary opposition’ to any purported “‘complete-ness’” of this new, two-dimensional ‘number-plane’, might involve an internal breach, or ‘immanent transcendence’, of some ‘number-definitional’ deficits owing to, e.g., the “Complex” numbers’ mere two-dimensionality.

The $\overset{3}{H}_{\#}^C$ “Complex numbers” -- or “the Complexes”, for short -- are, each and all, 2-D “non-amalgamative sums” [cf. Dr. Charles Musès], of the [more fully-explicit] form $ar + bri$, of integral, rational, or irrational quantifications of two qualitatively distinct unit-qualifiers, of two different kinds of numbers-units, each with its own 1-D ‘number-axis’, namely: the unit $r \equiv +1$, and/versus the unit $+ \sqrt{-1} \equiv +i$. These sums are “non-amalgamative” sums, precisely because such sums are qualitatively heterogeneous; because $+i \nmid +1 \equiv r$, and, therefore, because $\forall a, b \in \mathbb{R}, bi \equiv bri \nmid ar \equiv a$, just as any quantity of “apples” is qualitatively inhomogeneous with respect to any quantity of “oranges”, at least at the ‘meta-fractal scale/level’ of their ‘species» sub-categorical-units, «aufheben»-related to their ‘genos» category-unit of “fruit” as their ‘meta-«monad»-ic meta-category-unit. The $\overset{3}{H}_{\#}^C$ number-space is a ‘unitation’ of the “Reals”, \mathbb{R} , of the ‘imaginaries’ subset i , & of the ‘i-izations’ of every “inherited” component of the \mathbb{R} space. The arithmetic system $\overset{3}{H}_{\#}^C$ is $\overset{3}{H}_{\#}^i \otimes \overset{3}{H}_{\#}^R$, the ‘ $\overset{3}{H}_{\#}^i$ -ization’, or ‘aufheben’ of $\overset{3}{H}_{\#}^R$, ‘self-negation/-elevation’, of $\overset{3}{H}_{\#}^R$, in all its components, component-by-component, already including the “‘evolute’” [double-]«aufheben» conservation of $\overset{3}{H}_{\#}^R$.

The opposite kind-of-number(s) to the $\overset{3}{H}_{\#}^C$ kind, would therefore evidently be a [sub-]system-category of ‘non-2-D-numbers’ in some sense. We identify/interpret these ‘non-2-D-numbers’, with/as the [sub-]system of what we call the ‘hamilton numbers’, that we [may] have already found extant in our perhaps unsystematic, “chaotic” [Marx] knowledge/experience of the totality of the modern/contemporary ‘Standard Numbers’: the kind of number(s) that we denote herein by:

$$\overset{3}{H}_{\#}^h \mapsto \mathbb{H}_{64}$$

We have thus solved for the meaning of the ‘contra-thesis’ category of the i subset of the “imaginary” numbers ‘contra-thesis’ category itself --

$$\Delta(\overset{3}{H}_{\#}^i) = \overset{3}{H}_{\#}^i \oplus \overset{3}{H}_{\#}^i; \vdash \Delta(\overset{3}{H}_{\#}^i) = \overset{3}{H}_{\#}^h$$

We call the $\overset{3}{H}_{\#}$ *numbers* the ‘*h*amilton numbers’, for historical reasons, because the *j* and *k* units that ‘*uni*[t-i]fy’, respectively, the two mutually-perpendicular ‘number-dimensions’ of *h* were first discovered by Sir William Rowan *H*amilton, in his search for a “*three*-dimensional complex number”, that led, instead, to a “*four*-dimensional complex number, the “*quaternion*”: $ar + brj + crj + drk$, the generic element of the $\overset{3}{H}_{\#}$ system, generated by $\overset{3}{H}_{\#} \otimes \overset{3}{H}_{\#}$.

Strictly in terms of our “‘analytic-geometric’” vision/visualization of the Standard Arithmetics, first addressed in section *B.η.*, the emergence of the *a* from out of the self-critique of the *N* filled in a slot to the left of all of the “*Natural*” Numbers “points”, and the fruition of the *self-critique* of $\overset{3}{H}_{\#} \overset{a}{\#}$ as $\overset{3}{H}_{\#} \overset{m}{\#}$, marked a “filling-in” of a potentially-infinite ‘counter-ray’ of ‘number-“points”’, again all to the left of all of the “*Natural*” Numbers “points” ray. These first two movements, from the *N* to the *W*, and then from the *W* to the *Z*, both involve an “‘*extensive* filling-*in*’” -- in the form of a “‘filling-*out*[ward]’” -- of number-space.

The emergence of the *f* from out of the self-critique of the *m* marked the onset of a new phase; one also consisting of two consecutive movements -- first, from the *Z* to the *Q*, and, second, from the *Q* to the *R*, the latter having the *d* as their *limen* -- a phase of “‘*intensive* filling-*in*’”, in which new ‘number-“points”’ are added-*in*, *in-between* every pair of the ‘number-“points”’ previously ‘*extantized*’ explicitly, or ‘*outed*’, for the *Z*, and, then, next, even more so, for the *Q*.

In particular, the transition from the *Q* to the *R*, mediated and effected by $\overset{3}{H}_{\#} \overset{d}{\#}$, packs new ‘number-“points”’ “*continuously*”, among those ‘*Q* *whole*-number-“points”’, *whole*-number-“points” that have nothing other than a potentially-infinite succession of *0*s to the right of their *d*ecimal points in their *d*ecimal representations. The harbinger of this transformation, $\overset{3}{H}_{\#} \overset{d}{\#}$, in itself merely posits this “*continuity*” for the *RHS*, ‘positive-signed ray’ of the *Q*-numbers’ “line”-space.

But the advent of the *d*, nevertheless, already portends the comprehensive ‘*continuity*’ and ‘*potentially-infinite-d*ecimal-ization’ of this single “number-line” entire, negative, neutral, and positive alike, that arrives in the next *stage/step* of our ‘meta-model’, *stage/step* *S* = 5. In any case, the only at last ‘*numbers-dense*’ -- but all along and still vastly ‘por[e]-ous’ -- ‘*pseudo-solid*’ “line” of numbers that pertained from the *N* to the *W* to the *Z* and, at last, even to the *Q*, is, already, “here”, in *stage/step* *S* = 4, even with its merely “‘*formal* *subsumption*’” of $\overset{3}{H}_{\#} \overset{Q}{\#}$ by $\overset{3}{H}_{\#} \overset{d}{\#}$, on the *liminal* verge of becoming a “*Real*”, “*solid*”, “*line*” -- a “*continuous*”, rectilinear, *1-D* expanse ‘of [number-]points’, *without any more lacunae*, at least not from the $\overset{3}{H}_{\#} \overset{d}{\#}$ viewpoint -- or from the viewpoint of the “*Standard*” $\overset{3}{H}_{\#} \overset{R}{\#}$.

With the advent of the *i* increment of number ‘ideo-ontology’, a radically new mode of continuance of this continuing process of the “filling-in” of “‘number-space’” with ever new *kinds* of ‘number-points’ is born. The progressive “filling-*in*” of a single “number-*line*” ‘number-dimension’, all the way up to the point of “‘*solidity*’”, suddenly “‘morphs’” into the addition, whole, of an entire new, equally “‘*solid*’”/“‘*continuous*’”, *second* “number-*line*” ‘number-dimension’, perpendicular to the first. Does the *h* kind of number signify a yet further *extension* of this “adding whole new dimensions of number” m.o.?

The Equation, “‘Well-Formed’” within $\overset{3}{H}_{\#} \overset{C}{\#}$, that is *Not Solvable* within $\overset{3}{H}_{\#} \overset{C}{\#}$, and that Therefore Points, from within $\overset{3}{H}_{\#} \overset{C}{\#}$, to beyond $\overset{3}{H}_{\#} \overset{C}{\#}$ -- to within $\overset{3}{H}_{\#} \overset{H}{\#}$.

In terms of an immanent critique, or self-critique, of the “*Complex*” system of arithmetic, and, more specifically, in terms of the self-critique of the “*i*maginary” numbers [sub-]system of arithmetic, we have noted a family of algebraic equations, “‘well-formed’” within the algebra of the “*Complex*” numbers arithmetic, which are, however, *not* “satisfiable” within the “*Complex*” system -- which are *not* “solvable” by any “*Complex*” number but $0r + 0i = 0$. This ‘equations-family’ can be represented, generically, by the single equation, $[+x_{\zeta}y_{\zeta} = -y_{\zeta}x_{\zeta} \mid x_{\zeta}, y_{\zeta} \neq 0]$. A specific instance of this equations-family is $+iy_{\zeta} = -y_{\zeta}i$. This equation asserts a paradox for a definition of ‘Standard Number’ restricted to the concept of the “*Complex*”: ‘*the paradox of anti-commutativity*’, a form of multiplicative *non*-commutativity, in which the value of the product of certain pairs of numbers is ‘arithmetically negated’ -- is reversed in sign -- if their positions in the product-expression are interchanged.

Therefore, the self-critique of the “*Complexes*”, as seeded by the equation analyzed above, and, more specifically, the self-critique of the ‘*i*maginary numbers’, leads us to the ‘*h*amilton numbers’, $\overset{3}{H}_{\#} \overset{H}{\#} = \overset{3}{H}_{\#} \overset{h}{\#}$, the category of *numbers* that adds not one but *two* whole new, additional “*continuous*” ‘number-dimensions’ to our

developing “number-space”, which thus becomes not a *3-D* but a *4-D* “‘number-space’”. The *i* kind-of-numbers, in this *stage*’s now “‘*formal* *subsumption*’”

of/contrast with the *d* kind & their [sub-]system of arithmetic, $\overset{3}{H}_{\#} \overset{d}{\#}$, and their fullness as the *R* kind-of-number(s), and their system of arithmetic, $\overset{3}{H}_{\#} \overset{R}{\#}$, and in terms of our “‘analytic geometric’” visualization of these “number-spaces”, redefine ‘Standard Number’ to be, not just “‘counts’”, or at least ‘a-mounts’, in either of the two directions/“rays” inherent in a single “line”, a single “dimension” of number-*kind*, but also “‘counts’”, or at least ‘a-mounts’, along either of the two directions/rays inherent in a second “line”, a second “dimension”, of number-*kind*, orthogonal to the first. This *stage* of our ‘self-dialogue’ has therefore far *far* further “filled-in” our “analytic-geometric” number-space. It has equipped us conceptually with the inkling of a whole *new second linear number-space*, right after -- upon the very heels of -- our apparent completion of *the first line-ar number space*, and thus, given combinations/interactions between the two [sub-]spaces, more than doubling the *content* of our developing “number-space”. Via the advent of $\overset{3}{H}_{\#} \overset{h}{\#}$, the domain of number is now ‘re-seen’, in *this new relative light*, as *multi-dimensional* rather than as *merely one-dimensional*, or even as *merely two-dimensional*.

Thus, the **six** preceding horizons of our number-concept -- namely, those of the **N**, the **W**, the **Z**, the **Q**, the **R**, and the **C**, are, together, [re-]visualized, in the perspective opened via the vantage-point of $\frac{3}{H}h_{\#}$, as but **six** successive **s**tages in the construction of merely *the first two dimensions* of an at least **four-dimensional space** of number-**kinds**.

Note again that the Standard “‘number-space’” that succeeds the **two-dimensional** Standard “‘number-space’” of **C** is not a **three-dimensional** “‘number-space’”. It is a **four-dimensional** “‘number-space’”: **H**.

After the ‘analytic-geometric representation of **C** numbers’ emerged in the work of Wallis, de Moivre, Euler, Wessel, Argand, Gauss, and others, up into the early **1800s**, and proved efficacious for ‘arithmeticizing and algebraicizing the geometrics’ of the operations of directed line-segments, modeling, e.g., Newtonian velocities, momenta, accelerations, and forces confined to a plane-space, mathematicians began searching for a **three-dimensional** analogue of the **two-dimensional ‘complexes’**, **(x, y, i)**, to model, e.g., velocities, momenta, accelerations, and forces for physical three-space. However, finding familiar analogues for the operations of addition, subtraction, multiplication, division, exponentiation, etc., among such “‘3-tuples’”, or “‘triples’”, of coordinate values, **(x, y, z)** -- especially analogues concordant with the familiar commutative, associative, and distributive “‘laws’” -- proved daunting. Gauss arrived at a **non-commutative** “‘3-algebra’” in work dated **1819**, but did not publish it, perhaps because it fell short of what physicists needed. Hamilton too searched in vain for years for a “‘3-algebra’” that would also exhibit the kinds of operations and properties familiar from **R** and **C**. By **1843**, he had found he could make progress only by making two concessions to arithmetical ‘ideo-phenomena’ of unprecedented novelty, i.e., to new arithmetical ‘ideo-ontology’: **(1) multiplicative non-commutativity**, and; **(2) “‘number-space’” four-dimensionality**.

It is true that -- historically as an offshoot of the **quaternion** arithmetic -- an arithmetic of **vectors** was later discovered, that could confine itself, via its “‘cross product’” operation-rule, to a three-dimensional representation-space, principally through work by Maxwell, Heaviside, Gibbs, Grassmann, and others, work which, in effect, separated the “‘scalar” and “‘vector” components of Hamilton’s “‘quaternion product” into two separate operations of multiplication -- into a “‘scalar product” versus a “‘vector product”.

However, this vector arithmetic came at an enormous “‘price” in terms of new arithmetical ‘ideo-ontology’, including that these **new kinds of** [“‘vector”] **number(s)** exhibit the novel ‘ideo-phenomena’ of **(1) “‘nilpotent”** multiplication for the “‘vector product”, or “‘cross product”, operation-rule -- $\underline{v} \neq \underline{0}$, but $\underline{v} \times \underline{v} = \underline{0}$ -- plus of both **non-commutativity**, and **non-associativity**, and **absence of a division operation** as inverse to this multiplication operation, and; **(2) “‘zero divisor”** multiplication for the “‘scalar product” or “‘dot product”, rule, for mutually-perpendicular vectors -- $\underline{u}, \underline{v} \neq \underline{0}$, & $\underline{u} \perp \underline{v}$, means $\underline{u} \bullet \underline{v} = \underline{0}$ -- plus multiplicative **non-closure**, and ‘**non-trichotomy**’, i.e., the “‘dot” product involves a ‘**tetrachotomy principle**’, implying ‘**qualitative inequality among arithmetical values**’, in that the “‘dot” product of two vectors is **not** a vector, but a scalar -- $\underline{u} \bullet \underline{v} = \text{ar} \ \& \ \underline{u} \neq \text{ar} \ \& \ \underline{v} \neq \text{ar}$, i.e., a scalar value is an arithmetical value that is **qualitatively different**, not just **quantitatively different**, from a vector arithmetical value -- as well as the **absence of a division operation** as inverse to this multiplication operation.

Later, *circa* **1880**, F. Georg Frobenius and Charles Saunders Peirce proved, independently, a theorem stating that the only linear associative ‘division arithmetics’ with “‘**Real**” coefficients of the primary “‘imaginary” units, and with a finite number of primary “‘imaginary” units, an identity element for multiplication, and concordant with the “‘product law” [i.e., with a familiar “‘division operation”], are the arithmetics of the **R**, **C**, and **H** “‘number-spaces’”. In **1898**, Adolf Hurwitz proved that the **only** “‘normed” ‘division arithmetics’ [arithmetics having a unique positive “‘vector length” assigning function, i.e., a “‘norm”, for all of their values except their “‘zero vector(s)”] over the **R** “‘field” are those of the **R**, **C**, **H**, and **O** ‘number-kinds’, “‘up to isomorphism”. [for more about these issues, see, for example, Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford U. Press [NY: **1972**], vol. **2**, pp. **775-794**, & vol. **3**, pp. **1146-1157**].

Taking the Measure of Where We Have Arrived Cognitively via Step/Stage 6 in this Chain of Immanent Critiques.

Thus, we arrive at -- as the fruition of **Step/Stage 6** -- the ‘mutual controversion’ of $\frac{3}{H}c_{\#}$ vs. $\frac{3}{H}h_{\#}$. We have $\frac{3}{H}c_{\#}$ as the partial, one-sided, unnecessarily restrictive view of modern ‘Standard Number’ as limited to the “‘**continuous co-counting**”, the “‘**bi-counting**” of, and in, **units of the square root of negative “‘Real” unity**, as well as the “‘**counting**” of, and in, **units of positive “‘Real” unity**. And we have this $\frac{3}{H}c_{\#}$ in an ‘additively oppositional’ relation to $\frac{3}{H}h_{\#}$ as **counter-example** to $\frac{3}{H}c_{\#}$ ’s ‘number as “‘**continuous co-counting of Real unity and of Imaginary unity**”’ view. That is, we have $\frac{3}{H}h_{\#}$ as an axioms-sub-system of ‘**doubly-perpendicularly-transcending-complex-plane-number-“points”**’.

However, in itself, $\frac{3}{H}h_{\#}$ is just as “‘one-sided”/“‘partial” -- just as incomplete/inadequate -- as $\frac{3}{H}c_{\#}$, if not even more so. The $\frac{3}{H}h_{\#}$ axioms-[sub-]system of arithmetic encompasses **only j- & k-units**-based, **j-unit- & k-unit-counting** numbers, excluding/ignoring the entire **i-unit-ed- & -j-unit-ed**, **C[omplex] plane domain** of number-kind. So $\frac{3}{H}c_{\#}$ is equally a **counter-example** to $\frac{3}{H}h_{\#}$ -- to any claims of adequacy/completeness/totality that might be ‘mis-made’ on behalf of $\frac{3}{H}h_{\#}$.

We thus remain in an unsatisfactory situation, in terms of systematically -- including of “‘taxonomically” -- organizing & accounting for all of what our experience of the totality of modern, contemporary “‘Standard Arithmetic(s)” may include. To improve our situation further, we would have to move on, on to **Step/Stage 7**, to

$\frac{3}{H}H_{\#}^{\#} = \left(\frac{3}{H}N_{\#} \right)^2$, to the **self-«aufheben»**, i.e., to the **self-reflection**, or “‘**squaring with itself**”, of $\left(\frac{3}{H}c_{\#} \oplus \frac{3}{H}h_{\#} \right)$ [with] itself, or [shortcut] to the results of the action of ‘[de]flection’ upon the collective «**aufheben**» operator/operation denoted $\left(\frac{3}{H}c_{\#} \oplus \frac{3}{H}h_{\#} \right)$, by the «**aufheben**» operator/operation denoted $\frac{3}{H}h_{\#}$, i.e., to the results of the action of $\frac{3}{H}h_{\#}$ upon the entire, total result of **Step/Stage 6**.

What the Symbol $\overset{3}{\underset{H-\#}{C}}$ "Intends": *Axioms-System of the Arithmetic of the so-called "Complex" Numbers, C* [commenced]

C0. C is not the Empty Set. $C \neq \emptyset \equiv \{ \}$.

First Order Axioms of "Complex" Numbers: [mainly] "Phonogramic" Rendering: [mainly] "Ideogramic" Rendering:
CP1. 0 is a "Complex" Number. $0 \in C$;
CP2. The **successor** of any " R -like" "Complex" Number, c , is c . $c \in C_R \subset C \Rightarrow S(c) \in C$;
CP3. No two " R -like" "Complex" Numbers have the same **successor**. $S_{c_1}, S_{c_2} \in C_R \subset C \& c_1 \neq c_2 \Rightarrow S_{c_1} \neq S_{c_2}$;
CP4. 0 is **not** the **successor** of any " R -like" "Complex" Number. $\neg \exists c \in C_R \subset C [S(c) = 0]$. 0 has a **predecessor** in $C_R \subset C$, and that **predecessor** is $-1r + 0i$;
Second Order Mathematical Induction Axiom for "Complex" Numbers: [mainly] "Phonogramic" Rendering: [mainly] "Ideogramic" Rendering:
CP5.a. $\forall S [S \subset C \& 0 \in S \& [\forall c \in S, S(c) \in S] \Rightarrow S = C_R]$ [Induction along the C_R -axis of the "Complex" Plane];
CP5.b. $\forall S [S \subset C \& 0 \in S \& [\forall c \in S, S(c) \in S] \Rightarrow S = C]$ [Induction along the C -axis of the "Complex" Plane].

First Order Axioms of "Complex" Numbers Addition:

- CA1.** For all c_1, c_2 in C , $c_1 + c_2 = c_2 + c_1$ [Additive Commutativity];
CA2. For all c_1, c_2, c_3 in C , $(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3)$ [Additive Associativity];
CA3. For all c_1, c_2 in C , $c_1 + c_2$ is in C [Additive Closure of C]; $[\forall c_1, c_2 \in C][c_1 + c_2 \in C]$;
CA4. There is an element 0 in C such that, for all c in C , $0 + c = c$ [Additive Invariance Element];
CA5. For every c in C , there exists $(-c)$ in C , such that $c + (-c) = 0$ [Additive Inverse Elements];
 $[\forall c \in C][\exists -c \in C][c + (-c) = (-c) + c = 0]$.

First Order Axioms of "Complex" Numbers Multiplication:

- CM0.** There is an element 0 in C such that, for all c in C , $0 \times c = 0$ [Multiplicative Predominance];
 $[\exists 0 \in C][\forall c \in C][0 \times c = c \times 0 = 0]$;
CM1. For all c_1, c_2 in C , $c_1 \times c_2 = c_2 \times c_1$ [Multiplicative Commutativity];
CM2. For all c_1, c_2, c_3 in C , $(c_1 \times c_2) \times c_3 = c_1 \times (c_2 \times c_3)$ [Multiplicative Associativity];
CM3. For all c_1, c_2 in C , $c_1 \times c_2$ is in C [Multiplicative Closure of C]; $[\forall c_1, c_2 \in C][c_1 \times c_2 \in C]$;
CM4. There is an element 1 in C such that, for all c in C , $1 \times c = c$ [Unique Multiplicative Invariance Element];
CM5. For every $c \neq 0$ in C , there exists $(1/c)$ in C , such that $c \times (1/c) = 1$ [Multiplicative Inverse Elements];
 $[\forall c \in C][\exists 1/c \in C][c \times (1/c) = (1/c) \times c = 1]$; the value of the expression $(c/0)$ is undefined;

What the Symbol $\overset{3}{\underset{H-\#}{C}}$ "Intends" [continued & concluded].

First Order Axioms for the "Hybridization" of the "Complex" Numbers Operations of "Multiplication" and of "Addition":

- CH1.** For all c_1, c_2, c_3 in C , $c_1 \times (c_2 + c_3) = c_1 \times c_2 + c_1 \times c_3$ [Distributivity ["Hybridization"] of Multiplication over Addition];
CH2. For all c_1, c_2, c_3 in C , $(c_1 + c_2) \times c_3 = c_1 \times c_3 + c_2 \times c_3$ [Distributivity ["Hybridization"] of Addition over Multiplication];

Axioms of "Complex" Numbers Exponentiation:

- CE0.** For all c in $C - \{0\}$, $c^0 = 1$; the value of the expression 0^0 is undefined [Exponentiation Predominance];
CE1. For all $c_1, c_2, c_3 \neq 0$ in C_R , $c_1^{c_2} \times c_1^{c_3} = c_1^{c_2 + c_3}$ [using "principal value" rules] [Exponentiation Additivity] [$\subset \Delta R$];
CE2. For all $c_1, c_2 \in C_R$, $c_3 \neq 0$ in C , $(c_1^{c_2})^{c_3} = c_1^{c_2 \times c_3}$ [using "principal value" rules] [Exponentiation Multiplicativity] [$\subset \Delta R$];
CE3. For all $c_1, c_2, c_3 \neq 0$ in C_R , $(c_1 \times c_2)^{c_3} = c_1^{c_3} \times c_2^{c_3}$ [using "principal value" rules] [Exponentiation Distributivity] [$\subset \Delta R$];
CE4. For all $c_1, c_2 \neq 0$ in C , $c_1^{c_2}$ [using "principal value" rules] is in C [Exponentiation Closure] [$\subset \Delta R$];
CE5. For all c in C , $c^1 = c$ [Exponentiation Invariance];
CE6. For all $c_1, c_2, c_3 \neq 0$ in C_R , $(c_2/c_1)^{c_3} = (c_2/c_1)^{c_3} = c_2^{c_3}/c_1^{c_3}$; $(c_3/c_2)^{c_1} = c_3^{c_1}/c_2^{c_1}$ [Exponentiation Complex-ity] [$\subset \Delta R$];

Axioms of "Complex" Numbers Order:

- CO1.** For all c_1, c_2 in C , either $c_1 > c_2$, or $c_1 = c_2$, or $c_1 < c_2$, or $c_1 \nlessdot c_2$ [Tetrachotomy "Law"] [$\subset \Delta R$];

- CO2.** For all $c_1 > 0, c_2 > 0$ in C_R , $c_1 + c_2 > c_1, c_1 + c_2 > c_2$ [Additive Order for "Positive" R -like Complexes] [$\subset \Delta R$];
CO3. For all $c_1 > 1, c_2 > 1$ in C_R , $c_1 \times c_2 > c_1, c_1 \times c_2 > c_2$ [Multiplicative Order for "Positive" R -like Complexes] [$\subset \Delta R$];
CO4. Any non-empty set of "Complex" Numbers contains a **least** element [Well Ordering Principle [Second Order]]; $[\forall S \subset C][S \neq \emptyset][\exists s \in S][\forall t \in S][s \leq t] \Rightarrow s \in S$ [$\subset \Delta R$];
CO5. $0r + 0i < 1r + 0i$ [Order of Invariance Elements [First Order]];

Second Order Axiom of "Complex" Numbers Rationality:

- C1.** For every c in C , there is a c_r in the C -subset of " R -like complex numbers", $C_R \subset C$, such that $c_r \times c \in C_R \& [\exists c_r \in C_R][\forall c_r \in C_R][\exists c_r \in C_R][c_r \times c \in C_R]$ [$\subset \Delta R$];

Second Order Axiom of the Two-Dimensional "Continuousness" of the "Complex" Numbers [Plane]:

- CC1.a.** Every Set $S_a \neq \emptyset$ of " R -like complex" Numbers that has an **upper bound** also has a **least upper bound** [\equiv a "**supremum**"];
 $[\forall S_a][S_a \subset C_R][S_a \neq \emptyset][\exists A \in C_R][\forall s \in S_a][\forall a \in A][s < a] \Rightarrow [\exists a_{\min} \in A][\forall a \in A][a_{\min} \leq a]$ [$\subset \Delta R$];
CC1.b. Every Set $S_b \neq \emptyset$ of " I -like complex" Numbers that has an **upper bound** also has a **least upper bound** [\equiv a "**supremum**"];
 $[\forall S_b][S_b \subset C_I][S_b \neq \emptyset][\exists B \in C_I][\forall s \in S_b][\forall b \in B][s < b] \Rightarrow [\exists b_{\min} \in B][\forall b \in B][b_{\min} \leq b]$ [$\subset \Delta R$];

Axioms for the "Imaginary" Numbers [Line] among [within] the "Complex" Numbers [Plane]:

- CI1.** There is an "imaginary" number i in C_I in C , such that $i \times i = -1$ [Imaginary Unity] [$\subset \Delta R$];
CI2. For every c in C , there exist a, b in C_R in C , such that $c = a + (b \times i)$ [C -Complex Numbers' Two Parts];
 $[\forall c \in C][\exists a, b \in C_R \subset C][c = a + (b \times i)]$ [$\subset \Delta R$].

C.3. The Solution/Presentation as a Whole.

So, in summary, the following is our completed solution of the [E.D.](#) Dialectical Equation for the Dialectical, [Meta-]Systematic Presentation of the Systems-Progression of the axioms-systems of the Standard Arithmetics, through step/stage 6 of that Presentation, with the meanings that we have either taken as “givens”, or established ‘organonically’, above, for each ‘connotogramic’ term of the following **64**-term axioms-systems qualifiers-sum, wherein we are using ‘ \hookrightarrow ’ as the ‘assignment sign’, or ‘interpretation sign’, that asserts an association between a ‘*gene*’-ric qualifiers-sum & a «*speci*»-fic [*speci*fically-interpreted] -- ‘qualifiers-sum’:

$$\begin{aligned}
 \underline{\text{H}}_{\text{h}} = 6 &= \left[\left[\text{L}_1 \right] \right]^{2^6} = \left[\left[\text{L}_1 \right] \right]^{64} = \\
 &\text{L}_1 \text{ L}_2 \text{ L}_3 \text{ L}_4 \text{ L}_5 \text{ L}_6 \text{ L}_7 \text{ L}_8 \text{ L}_9 \text{ L}_{10} \text{ L}_{11} \text{ L}_{12} \text{ L}_{13} \text{ L}_{14} \text{ L}_{15} \text{ L}_{16} \\
 &\text{L}_{17} \text{ L}_{18} \text{ L}_{19} \text{ L}_{20} \text{ L}_{21} \text{ L}_{22} \text{ L}_{23} \text{ L}_{24} \text{ L}_{25} \text{ L}_{26} \text{ L}_{27} \text{ L}_{28} \text{ L}_{29} \text{ L}_{30} \text{ L}_{31} \text{ L}_{32} \\
 &\text{L}_{33} \text{ L}_{34} \text{ L}_{35} \text{ L}_{36} \text{ L}_{37} \text{ L}_{38} \text{ L}_{39} \text{ L}_{40} \text{ L}_{41} \text{ L}_{42} \text{ L}_{43} \text{ L}_{44} \text{ L}_{45} \text{ L}_{46} \text{ L}_{47} \text{ L}_{48} \\
 &\text{L}_{49} \text{ L}_{50} \text{ L}_{51} \text{ L}_{52} \text{ L}_{53} \text{ L}_{54} \text{ L}_{55} \text{ L}_{56} \text{ L}_{57} \text{ L}_{58} \text{ L}_{59} \text{ L}_{60} \text{ L}_{61} \text{ L}_{62} \text{ L}_{63} \text{ L}_{64} \\
 \hookrightarrow & \\
 \text{H}_{\text{H}}^{\#} &= \left(\text{H}_{\text{H}}^{\#} \right)^{2^6} = \left(\text{H}_{\text{H}}^{\#} \right)^{64} = \\
 \text{H}_{\text{H}}^{\#} \oplus \text{H}_{\text{H}}^{\#} & \\
 \oplus \text{H}_{\text{H}}^{\#} \text{a} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{m} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{ma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{ma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{f} \text{N} \oplus \\
 \text{H}_{\text{H}}^{\#} \text{f} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{fa} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{fa} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{fm} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{fm} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{fma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{fma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{d} \text{N} \oplus \\
 \text{H}_{\text{H}}^{\#} \text{d} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{da} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{da} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dm} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dm} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{df} \text{N} \oplus \\
 \text{H}_{\text{H}}^{\#} \text{df} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dfa} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dfa} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dfm} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dfm} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dfma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{dfma} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{i} \text{N} \oplus \\
 \text{H}_{\text{H}}^{\#} \text{i} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{ia} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{ia} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{im} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{im} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{ima} \text{N} \oplus \text{H}_{\text{H}}^{\#} \text{ima} \text{N}
 \end{aligned}$$

[illegible]

$3\text{H}_{\#} \text{Z}_{\#} \text{---} \oplus \text{---} 3\text{H}_{\#} \text{f}_{\#}$
 $\oplus 3\text{H}_{\#} \text{fN} \# \oplus 3\text{H}_{\#} \text{fa} \# \oplus 3\text{H}_{\#} \text{faN} \# \oplus 3\text{H}_{\#} \text{fm} \# \oplus 3\text{H}_{\#} \text{fmN} \# \oplus 3\text{H}_{\#} \text{fma} \# \oplus 3\text{H}_{\#} \text{g}_{\#} \text{---} \oplus \text{---} 3\text{H}_{\#} \text{d}_{\#}$
 $\oplus 3\text{H}_{\#} \text{dN} \# \oplus 3\text{H}_{\#} \text{da} \# \oplus 3\text{H}_{\#} \text{daN} \# \oplus 3\text{H}_{\#} \text{dm} \# \oplus 3\text{H}_{\#} \text{dmN} \# \oplus 3\text{H}_{\#} \text{dma} \# \oplus 3\text{H}_{\#} \text{dmanN} \#$
 $\oplus 3\text{H}_{\#} \text{df} \# \oplus 3\text{H}_{\#} \text{dfN} \# \oplus 3\text{H}_{\#} \text{dfa} \# \oplus 3\text{H}_{\#} \text{dfaN} \# \oplus 3\text{H}_{\#} \text{dfm} \# \oplus 3\text{H}_{\#} \text{dfmN} \# \oplus 3\text{H}_{\#} \text{dfma} \#$
 $\oplus 3\text{H}_{\#} \text{dfmaN} \# \oplus 3\text{H}_{\#} \text{i}_{\#} \oplus 3\text{H}_{\#} \text{iN} \# \oplus 3\text{H}_{\#} \text{ia} \# \oplus 3\text{H}_{\#} \text{iaN} \# \oplus 3\text{H}_{\#} \text{im} \# \oplus 3\text{H}_{\#} \text{imN} \# \oplus 3\text{H}_{\#} \text{ima} \#$
 $\oplus 3\text{H}_{\#} \text{imaN} \# \oplus 3\text{H}_{\#} \text{if} \# \oplus 3\text{H}_{\#} \text{ifN} \# \oplus 3\text{H}_{\#} \text{ifa} \# \oplus 3\text{H}_{\#} \text{ifaN} \# \oplus 3\text{H}_{\#} \text{ifm} \# \oplus 3\text{H}_{\#} \text{ifmN} \#$
 $\oplus 3\text{H}_{\#} \text{ifma} \# \oplus 3\text{H}_{\#} \text{ifmaN} \# \oplus 3\text{H}_{\#} \text{id} \# \oplus 3\text{H}_{\#} \text{idN} \# \oplus 3\text{H}_{\#} \text{ida} \# \oplus 3\text{H}_{\#} \text{idaN} \#$
 $\oplus 3\text{H}_{\#} \text{idm} \# \oplus 3\text{H}_{\#} \text{idmN} \# \oplus 3\text{H}_{\#} \text{idma} \# \oplus 3\text{H}_{\#} \text{idmaN} \# \oplus 3\text{H}_{\#} \text{idf} \# \oplus 3\text{H}_{\#} \text{idfN} \#$
 $\oplus 3\text{H}_{\#} \text{idfa} \# \oplus 3\text{H}_{\#} \text{idfaN} \# \oplus 3\text{H}_{\#} \text{idfm} \# \oplus 3\text{H}_{\#} \text{idfmN} \# \oplus 3\text{H}_{\#} \text{idfma} \# \oplus 3\text{H}_{\#} \text{idfmaN} \# \oplus 3\text{H}_{\#} \text{h}_{\#}$

$^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{d}$ $\text{H}\text{--}\text{d}$
 $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{da}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{daN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dm}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dmN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dma}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dmaN}$
 $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{df}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dfN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dfa}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dfaN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dfm}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dfmN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dfma}$
 $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{d}$ $\text{H}\text{--}\text{d}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{diN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dia}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{diaN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dim}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dimN}$ $^3\text{H}\text{--}\text{Q}\text{--}\text{H}\text{--}\text{dima}$

[illegible]

$$\begin{aligned} \mathbb{H}_{\#}^3 \mathbf{N} &\rightarrow \mathbb{H}_{\#}^3 \mathbf{N}^2 = (\mathbb{H}_{\#}^3 \mathbf{N} \oplus \mathbb{H}_{\#}^3 \mathbf{a}) \rightarrow (\mathbb{H}_{\#}^3 \mathbf{N} \oplus \mathbb{H}_{\#}^3 \mathbf{a})^2 = (\mathbb{H}_{\#}^3 \mathbf{W} \oplus \mathbb{H}_{\#}^3 \mathbf{m}) \rightarrow \\ (\mathbb{H}_{\#}^3 \mathbf{W} \oplus \mathbb{H}_{\#}^3 \mathbf{m})^2 &= (\mathbb{H}_{\#}^3 \mathbf{Z} \oplus \mathbb{H}_{\#}^3 \mathbf{f}) \rightarrow (\mathbb{H}_{\#}^3 \mathbf{Z} \oplus \mathbb{H}_{\#}^3 \mathbf{f})^2 = (\mathbb{H}_{\#}^3 \mathbf{Q} \oplus \mathbb{H}_{\#}^3 \mathbf{d}) \rightarrow \\ (\mathbb{H}_{\#}^3 \mathbf{Q} \oplus \mathbb{H}_{\#}^3 \mathbf{d})^2 &= (\mathbb{H}_{\#}^3 \mathbf{R} \oplus \mathbb{H}_{\#}^3 \mathbf{i}) \rightarrow (\mathbb{H}_{\#}^3 \mathbf{R} \oplus \mathbb{H}_{\#}^3 \mathbf{i})^2 = (\mathbb{H}_{\#}^3 \mathbf{C} \oplus \mathbb{H}_{\#}^3 \mathbf{h}) \rightarrow \dots \end{aligned}$$

Summary & Prospect: Narrative Commentary on this Solution-Presentation as a Whole. As the immediately-above-rendered ‘qualifier-series’, ‘qualifiers-sum’, or ‘*cumulum*’, whose most compact canonical forms are --

$\overset{3}{H}\overset{\#}{C} \dashv \oplus \overset{3}{H}\overset{\#}{h}$, or $\overset{3}{H}\overset{\#}{H}_6$, or $\overset{3}{N}_2^6$ -- expresses, our solution-presentation is one which progresses in «*aporia*», that is, in ‘system-of-arithmetic-versus-counterexample’ pairs. The self-reflection, or immanent critique -- i.e., the self-critique -- of our «*arché*» system of arithmetic, $\overset{3}{H}\overset{\#}{N}$, yields our first ‘counter-system’, $\overset{3}{H}\overset{\#}{a} \equiv \overset{3}{H}\overset{\#}{a}$, which is also our first system-of-arithmetic-versus-counterexample pair/«*aporia*»: $\overset{3}{H}\overset{\#}{N} \dashv \oplus \overset{3}{H}\overset{\#}{a} \Leftrightarrow \overset{3}{H}\overset{\#}{a}_1 \boxplus \overset{3}{H}\overset{\#}{a}_2$.

From there on, every new ‘counter-exemplary counter[-sub]-system’ -- every new ‘contra-thesis’ category of arithmetic -- every new «*aporia*»-sum pairing, arises by the “‘self-reflection’” -- i.e., by the immanent critique -- of its immediate predecessor ‘counter-example’, ‘counter[-sub]-system’, ‘contra-thesis’ category, starting with $\overset{3}{H}\overset{\#}{aa} \equiv \overset{3}{H}\overset{\#}{m}$. And then, next, every such ‘contra-thesis’ category -- in the stage/step right after the one in which it first forms -- “‘conquers’”, “‘converts’”, “‘takes possession of’”, “‘appropriates’”, “‘assimilates’”, “‘uplifts’”/“‘elevates’”, “‘adjusts’”, or “‘combines with’” every previously-precipitated category, including itself, in forming its representation of the next full system of arithmetic, + that next system’s own new ‘contra-thesis’ counter-example counter[-sub]-system.

That is, in each phase, stage, or step of this ‘self-dialogic’ / ‘self-arguementing’ / ‘self-augmenting’ progression, for every such ‘contra-thesis’ category, it is none other than the ‘self-conquering’ -- the ‘self-conversion’, the immanently critical ‘self-possession’, the critical ‘self-appropriation’, the «*aufheben*» ‘self-assimilation’, the «*aufheben*» ‘self-elevation’, the ‘self-corrective self-adjustment’, the ‘meta-monadic self-combination’ -- of that ‘contra-category’ which creates, or concretizes, the next ‘counter-exemplary counter[-sub]-system’, the next ‘contra-thesis’ category, &, thereby, the next «*aporia*». This progression of «aporia» is potentially-infinite in Aristotle’s sense. That is, it has no known stage/step of intrinsic self-termination in terms of possibility. It thus exhibits a potential-infinity version of the Gödelian “*Inexhaustibility of Mathematics*”. However, actually, historically, to date, it does end -- e.g., as a widely-diffused human-phenomic achievement of human “universal labor” [Marx]. Presently, perhaps, it ends with $\overset{3}{H}\overset{\#}{G}$, the Grassmannian-dialectical, n-dimensional “‘arithmetic of geometries’”, in terms of that part of the potential infinity of possibility that Terran humanity has actualized, so far.

If we reflect upon the course of this ‘meta-systematic dialectical’ presentation of the standard arithmetics as a whole, in terms of its representation by the progression of “‘number-spaces’”, the progression of arithmetical analytical geometries by which we have ‘thematized’ it, we see the following. The first few phases -- ending in phases/stages 2 and 3 -- of this movement, *after* the initial positing of $\overset{3}{H}\overset{\#}{N}$, in stage 0, and its own immediate self-critique, in stage 1, yielding/constructing $\overset{3}{H}\overset{\#}{W}$, in stage 2, and including the self-critique of the «*aporia*» of $\overset{3}{H}\overset{\#}{W}$ and $\overset{3}{H}\overset{\#}{m}$, yielding/constructing $\overset{3}{H}\overset{\#}{Z}$, in stage 3, were, in terms of ‘numbers-spaces’, all about the “‘filling-out’”/“‘explicitization’” of “‘number-points’” to the left of the N “‘number-points’”, to the left side ‘lessor-side’ of the “‘number-point’” labeled I.

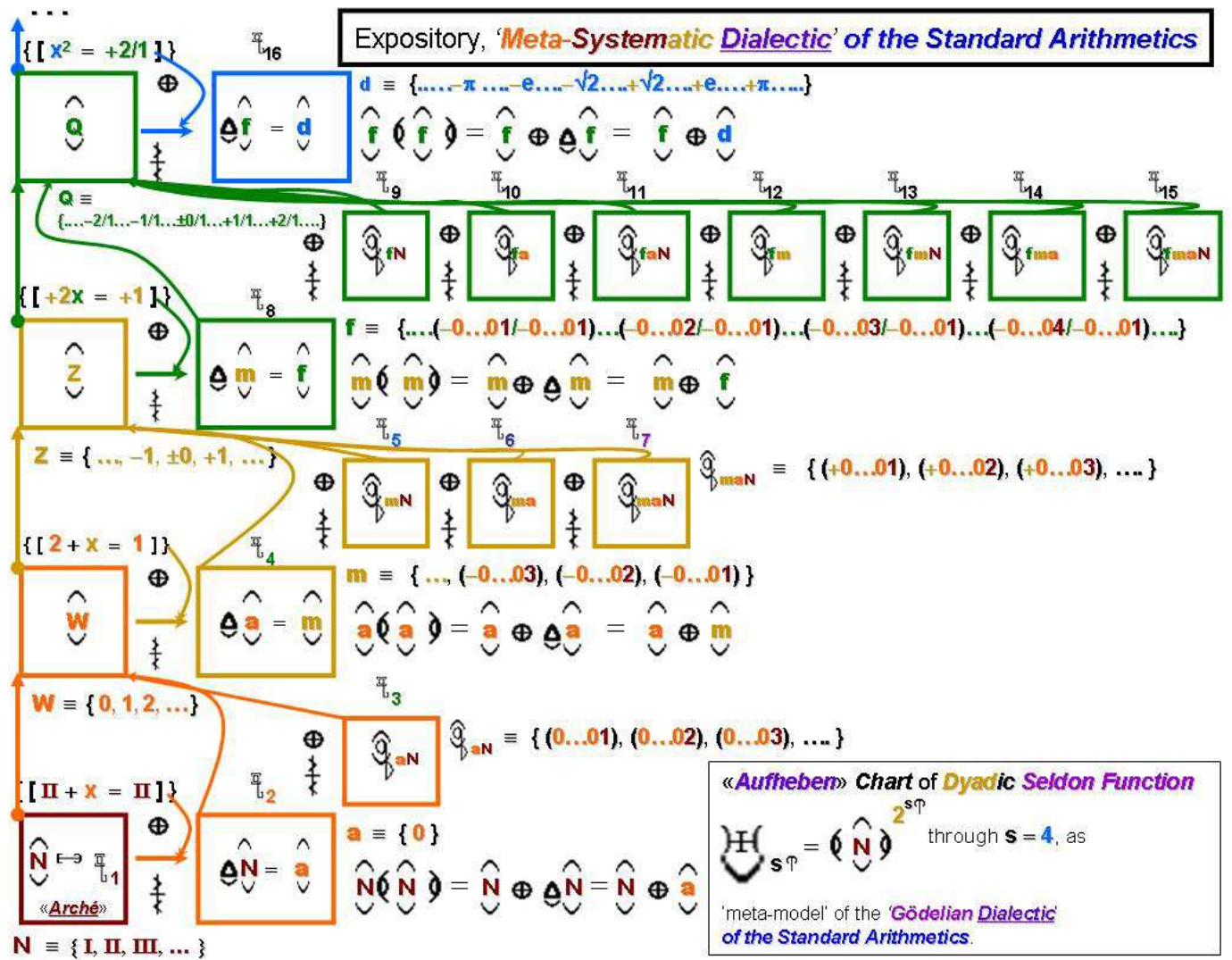
The next two phases of this movement, stages 4 and 5, were about “‘filling-in’”/“‘explicitizing’” new “‘number-points’”, in-between the Z ‘integ[er]al’ “‘number-points’”, via the self-critique of the $\overset{3}{H}\overset{\#}{Z}$ & $\overset{3}{H}\overset{\#}{f}$ «*aporia*» of stage 3, yielding/constructing the $\overset{3}{H}\overset{\#}{Q}$ & $\overset{3}{H}\overset{\#}{d}$ «*aporia*» of stage 4, followed by its own self-critique, yielding the $\overset{3}{H}\overset{\#}{R}$ & $\overset{3}{H}\overset{\#}{i}$ «*aporia*» of stage 5, and, in the process, first ‘densifying’, and then ‘solid-ifying’, the emergent, single, “solid” “number-line” analytical geometry of “Real” number.

The final phase of this movement covered by our presentation herein, that of the transition from stage 5 to stage 6, driven by the self-critique of the $\overset{3}{H}\overset{3}{R}_{\#}$ & $\overset{3}{H}\overset{3}{I}_{\#}$ «*aporia*», opened a new, and to-be-enduring, «*modus operandi*» of the ‘ideo-ontological extent-tion’ of number-*kinds* and of their arithmetics, no longer adding just new “points”, but evoking/‘explicitizing’ implied entire new “number-lines” -- whole new “‘dimensions’” of number(s) -- from out of their former invisible implicitude in “‘number-space’”. In the part of this story that we have recounted herein, this new “m.o.” is just starting, by the adjunction -- and by the “Cartesian product” interaction, with the “**Real**” ‘number(s)-dimension’, with the ‘**r**-axis’, with **R** -- of the so-called “**i**maginary” ‘number(s)-dimension’, of the “‘**i**-axis’”, of **i** -- the ‘number(s)-axis’ with **i** as its unit[y], a whole new *hyper*-“*dense*”, or “*solid*”, number-*line* -- to form the ‘number(s)-*plane*’ of the standard “**C**omplex Numbers”, $\mathbf{C} = \mathbf{C}^1 \sqsubset \overset{3}{H}\overset{3}{C}_{\#}$. Something of the further course of this new «*modus operandi*», in our interpretation of the *Gödelian Dialectic* ‘Meta-Equation’ beyond stages $\mathbf{s}_{\#} = \mathbf{0}$ through $\mathbf{s}_{\#} = \mathbf{6}$, recounted herein, can be summarized as follows. If we relegate/recede/demote the **4** ‘pre-**R**’ stages/steps of our ‘meta-systematic dialectical’ presentation -- i.e., those **4** «*aporia*» which first exhibit the “‘number-spaces’” of the $\overset{3}{H}\overset{3}{N}_{\#}$, $\overset{3}{H}\overset{3}{W}_{\#}$, $\overset{3}{H}\overset{3}{Z}_{\#}$, & $\overset{3}{H}\overset{3}{Q}_{\#}$ arithmetics -- to a kind of collective ‘sub-**Real**’ class, then we have $\overset{3}{H}\overset{3}{R}_{\#}$ as a kind of ‘neo-«*arché*»’, with the following arithmetic-systems succession, ‘thematized’ via adds of number-space axes:

W Dimensionality Range, Name, and Dimensionality Character of the **W**th “Number-Space”

- 0** $2^0 = 1 \hookrightarrow \mathbf{R}$: **R**eals; **1** single, *fixed* “continuous” “number-line” dimension/axis;
- 1** $2^1 = 2 \hookrightarrow \mathbf{C}$: **C**omplexes; **2** *fixed*, “continuous”, mutually-perpendicular “number-line” axes;
- 2** $2^2 = 4 \hookrightarrow \mathbf{H}$: **H**amilton Quaternions; **4** *fixed*, “continuous”, perpendicular “number-lines”/axes;
- 3** $2^3 = 8 \hookrightarrow \mathbf{O}$: **O**ctonions; **8** *fixed*, “continuous”, perpendicular “number-line” dimensions/axes;
- 4** $2^4 = 16 \hookrightarrow \mathbf{S}$: **S**edenions; **16** *fixed*, “continuous”, perpendicular “number-line” dimensions/axes;
- {n}** **N** $\hookrightarrow \mathbf{K}$: Clifford arithmetics; for any *fixed* **n** of “‘continuous, orthogonal number-lines’”; ...
- w^{\uparrow} **W** $\hookrightarrow \mathbf{G}$: **G**rassmann [nilpotent] numbers. Multiplication of **2** *distinct* **G**rassmann numbers of dimensionality **n** produces a **G**rassmann number of dimensionality **w + 1**; *variable, varying, escalating* dimensionality. Squaring a **G**rassmann number produces “nil”, e.g., $\underset{\cdot}{\cdot} \times \underset{\cdot}{\cdot} = \underset{\cdot}{\cdot}$.

- w^{\uparrow} **W** $\hookrightarrow w\mathbf{Q}$: Seldonian ‘Ontological-**Q**ualifier meta-numbers’. Self-multiplication of any single $w\mathbf{Q}$ ‘meta-number’ unit-**q**ualifier, $\overline{\mathbb{I}}_n$, represented analytical-geometrically as a unit-length, one-dimensional, ‘impartable’ directed line-segment, or ‘meta-vector’, yields $\overline{\mathbb{I}}_n + \overline{\mathbb{I}}_{2n} \mid n \in \mathbf{W} - \{0\}$, as its product, i.e., the **(+√2)**-length diagonal of the [2-dimensional] square parallelogram formed by the mutually-perpendicular ‘meta-vectors’ $\overline{\mathbb{I}}_n$ and $\overline{\mathbb{I}}_{2n}$, joined at their $\overline{\mathbb{I}}_0$ origin. If $\overline{\mathbb{I}}_1$, the «*arché*» of the $w\mathbf{Q}$ ‘meta-numbers’, is self-multiplied, and, successively, recursively, each product of such self-multiplication is, in its own turn, self-multiplied, as in the ‘Dyadic Seldon Function’, then this yields ‘[hypo-/hyper-]unit-cubic spaces’ of successive dimensionalities **1, 2, 4, 8, 16, 32, 64, ...**; *variable, self-varying, self-escalating* dimensionalities; able to model *dialectics*, e.g. the foregoing systems progression; ...



Interconnexion of the Dialectical Progression of the Standard Arithmetics with Other Core E.D. Dialectical Progressions.

While the dialectical equations-sequence captured by --

$$\underline{\mathbb{H}}_{\mathbb{H}}^{\#}([0, 6]) = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^{[0, 6]}}$$

-- represents an iterated, repeated, recurrent, immanent, seven-stage reflection upon, and immanent-critique/self-critique of, the Standard Arithmetics, it does not represent F.E.D.'s deepest, most searching immanent-/self-critique, of arithmetic(s)-in-general.

True, the [meta-]systematic dialectic of $\#$ modeled by 1., below --

$$1. \quad \underline{\mathbb{H}}_{\mathbb{H}}^{\#}([6]) = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^6}$$

-- elaborates ever more elaborately, as we have seen, but it never transcends the horizon of its «*arché*» concept of “cardinal” number”, or of “counting” number”, $\mathbb{H}_{\mathbb{H}}^{\mathbb{N}}$, albeit in an eventual context of ‘multi-dimensional counting’: of counting along multiple, qualitatively distinct [‘convolute-unit-qualifier-distinguished’] dimensions. A far deeper dialectical questioning of the concept of “number” requires a far deeper, far richer «*arché*»-thesis, and, hence, one with a far deeper, far richer ‘*first contra-thesis*’, or “‘*first counter-example*’”, axioms-[sub-]system of “‘number’”, with a far wider *chasm* between the 2, &, hence, with a much deeper initial «*aporia*», than that between “*counts* as *numbers*”, $\mathbb{H}_{\mathbb{H}}^{\mathbb{N}}$, & ‘*no[n]-counts* as *numbers*’, $\mathbb{H}_{\mathbb{H}}^{\mathbb{a}}$, in --

$$\underline{\mathbb{H}}_{\mathbb{H}}^{\#}([1]) = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^1} = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^2} = \mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right) = \hookrightarrow \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right) = \mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \multimap \mathbb{H}_{\mathbb{H}}^{\mathbb{a}}$$

-- as a dialectic within the “‘meta-system’” of higher-than-first-order axiomatic systems of arithmetic. That deeper dialectic is given as 2., below: ‘The Meta-Systematic Dialectic of the Axioms-Systems of the Dialectical Arithmetics’³, $\#$, as given by the Encyclopedia Dialectica ‘meta-model’ --

$$2. \quad \underline{\mathbb{H}}_{\mathbb{H}}^{\#}([s]_{\#}) = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^{s_{\#}}}$$

-- such that --

$$\underline{\mathbb{H}}_{\mathbb{H}}^{\#}([1]) = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^1} = \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right)^{2^2} = \mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right) = \hookrightarrow \left(\mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \right) = \mathbb{H}_{\mathbb{H}}^{\mathbb{N}} \multimap \mathbb{H}_{\mathbb{H}}^{\mathbb{Q}}$$

-- whose «*aporia*» is that of an arithmetic of “pure, unqualified ordinal quantifiers as *numbers*”, $\mathbb{H}_{\mathbb{H}}^{\mathbb{N}}$, vs. that of ‘pure, unquantifiable ordinal qualifiers as *meta-numbers*’, $\mathbb{H}_{\mathbb{H}}^{\mathbb{Q}}$, for first-order-only axioms-systems.

The “‘externalized’” ‘oppositional sum’ of the apparently ‘mutually external’ terms, $\mathbb{N}_{\#}$ and $\mathbb{Q}_{\#}$, namely, $\mathbb{N}_{\#} \multimap \mathbb{Q}_{\#}$, is founded, at root, upon what we term the ‘self-duality’ of $\mathbb{N}_{\#}$, i.e., in its immanent, internal, qualitative ‘self-opposition’, or ‘self-antithesis’: ‘quantitative ordinality \multimap qualitative ordinality’.

As this essay is intended, in part, to demonstrate/exemplify, it is the case that the $\mathbb{Q}_{\#}$ dialectical arithmetic forms a fundamental supplement, and ‘supplementary opposite’, to the $\mathbb{N}_{\#}$ arithmetic, to which $\mathbb{N}_{\#}$ is typically reduced. The $\mathbb{Q}_{\#}$ arithmetic instantiates, inculcates, and facilitates powerful, “‘purely-qualitative’”

cognitive capabilities that reside beyond the ken of the $\mathbb{N}_{\#}$ -- or “‘purely-quantitative’” -- capital-value «*mentalité*», the «*mentalité*» that, most of all, characterizes the “‘human phenome’” in general, including its partly consciously, *but mostly unconsciously* ideology-compromised sciences, during the capitalist epoch of the self-development of the human «*species*», under the unconscious cognitive dominion of the “law of [capital-]value” and its “elementary form of [commodity-]value” [Marx] as unconscious or semi-conscious universal paradigm. Of course, the way forward for humanity also involves transcendence of the «*aporia*» of $\mathbb{N}_{\#} \multimap \mathbb{Q}_{\#}$, to the system of their first ‘qualo-quantitative’ arithmetic as that «*aporia*»’s solution / dialectical synthesis, $\mathbb{U}_{\#}$, and beyond, to the seqq. systems of $\mathbb{U}_{\#}$, each one more determinate -- more ‘thought-concrete’ -- in its expressive power than its predecessor.

This deeper dialectic, one that encompasses the dialectic of number as «*arithmos*» in the ancient Hellenistic(⊕) sense², also intersects, or interconnects with, five other key *Encyclopedia Dialectica* dialectics of ‘The Human Phenome’ --

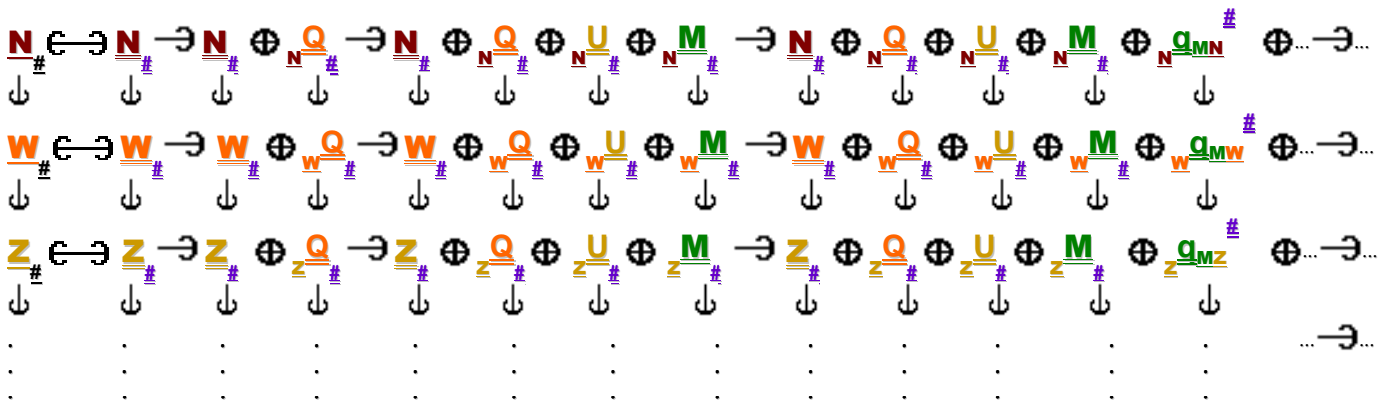
3. The Axioms-Systems Dialectic of Arithmetical Logics, \mathbf{L} , $\mathbf{L}^3_{\mathbf{H}, \mathbf{w}, \mathbf{E}}$ denotes Boolean Algebra’s ‘Logic-arithmetic’;
4. The [Psycho]Historical Dialectic of the “‘Meta-Evolution’” of Arithmetics, $\mathbf{\#}$.
5. The [Psycho]Historical Dialectic of Human-Social formation(s) “‘Meta-Evolution’”, \mathbf{f} .
6. The [Psycho]Historical Dialectic of Human-Social Relations of Production “‘Meta-Evolution’”, \mathbf{R} .
7. The [Psycho]Historical Dialectic of Human-Social Forces of Production “‘Meta-Evolution’”, \mathbf{F} .

You will find, rendered below, multi-directional systems-progression content-structures, which depict the interweaving of the first three of the four ‘Meta-Models’ that, together, form the core of the *Encyclopedia Dialectica* Immanent Critique Of Arithmetics [ICOA]. The following is a list of all seven ‘meta-models’ cited --

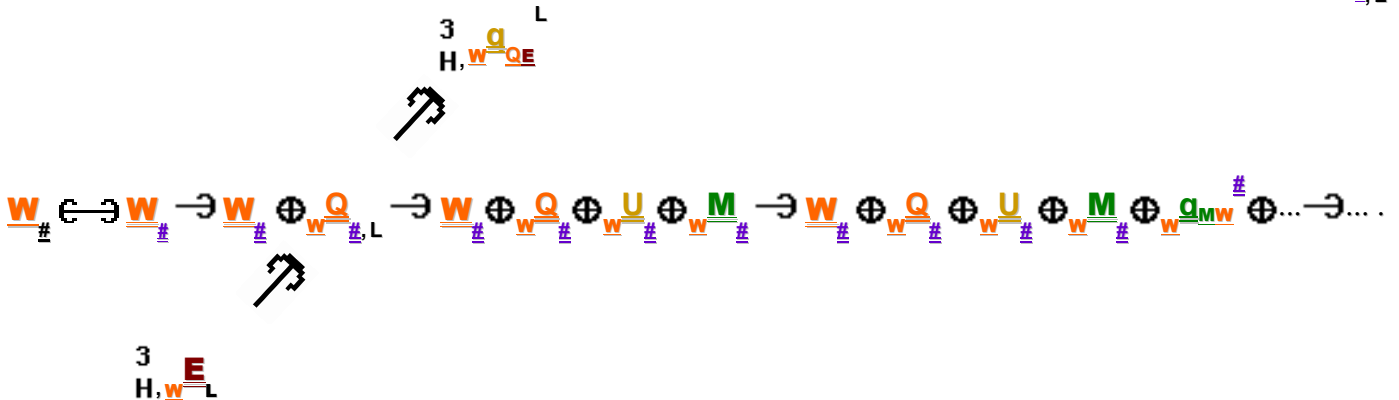
1.
$$\mathbf{3}_{\mathbf{H}} \mathbf{H} \left(\mathbf{s}_{\mathbf{\#}}^{\mathbf{\#}} \right) \uparrow = \left(\mathbf{3}_{\mathbf{H}} \mathbf{N}_{\mathbf{\#}} \right) \mathbf{2}^{\mathbf{s}_{\mathbf{\#}} \uparrow}$$
2.
$$\mathbf{3}_{\mathbf{H}} \mathbf{H} \left(\mathbf{s}_{\mathbf{\#}}^{\mathbf{\#}} \right) \uparrow = \left(\mathbf{3}_{\mathbf{H}} \mathbf{N}_{\mathbf{\#}} \right) \mathbf{2}^{\mathbf{s}_{\mathbf{\#}} \uparrow}$$
3.
$$\mathbf{3}_{\mathbf{H}} \mathbf{H} \left(\mathbf{s}_{\mathbf{L}}^{\mathbf{L}} \right) \uparrow = \left(\mathbf{3}_{\mathbf{H}} \mathbf{E}_{\mathbf{L}} \right) \mathbf{2}^{\mathbf{s}_{\mathbf{L}} \uparrow}$$
4.
$$\mathbf{2}_{\mathbf{h}} \mathbf{h} \left(\mathbf{\tau}_{\mathbf{\#}}^{\mathbf{\#}} \right) \uparrow = \left(\mathbf{2}_{\mathbf{h}} \mathbf{N}_{\mathbf{\#}} \right) \mathbf{2}^{\mathbf{\tau}_{\mathbf{\#}} \uparrow}$$
5.
$$\mathbf{2}_{\mathbf{h}} \mathbf{h} \left(\mathbf{\tau}_{\mathbf{f}}^{\mathbf{f}} \right) \uparrow = \left(\mathbf{2}_{\mathbf{h}} \mathbf{b}_{\mathbf{f}} \right) \mathbf{2}^{\mathbf{\tau}_{\mathbf{f}} \uparrow}$$
6.
$$\mathbf{2}_{\mathbf{h}} \mathbf{h} \left(\mathbf{\tau}_{\mathbf{R}}^{\mathbf{R}} \right) \uparrow = \left(\mathbf{2}_{\mathbf{h}} \mathbf{A}_{\mathbf{R}} \right) \mathbf{2}^{\mathbf{\tau}_{\mathbf{R}} \uparrow}$$
7.
$$\mathbf{2}_{\mathbf{h}} \mathbf{h} \left(\mathbf{\tau}_{\mathbf{F}}^{\mathbf{F}} \right) \uparrow = \left(\mathbf{2}_{\mathbf{h}} \mathbf{R}_{\mathbf{h}, \mathbf{F}} \right) \mathbf{2}^{\mathbf{\tau}_{\mathbf{F}} \uparrow}$$

The «*arché*» for Dialectical Equation 4., above, $\mathbf{2}_{\mathbf{h}} \mathbf{N}_{\mathbf{\#}}$, connotes Dr. Denise Schmandt-Besserat’s reconstruction of the practical, informal beginning of written numbers, $\mathbf{N} \equiv \{ \mathbf{\nabla}, \dots, \mathbf{\bullet}, \dots \}$, in ancient Babylonia. The «*arché*» for Dialectical Equation 5., $\mathbf{2}_{\mathbf{h}} \mathbf{b}_{\mathbf{f}}$, connotes the primordial, “bands” of predatory/foraging/scavenging/hunting-and-gathering ‘proto-human[oid]s’ who, to the best of our contemporary knowledge, constituted the ultimate ancestor-population of Terran humanity [see, for example, Robert Wright, *Non-Zero: The Logic of Human Destiny*, Pantheon, New York, 2000]. The «*arché*» for Dialectical Equation 6., $\mathbf{2}_{\mathbf{h}} \mathbf{A}_{\mathbf{R}}$, connotes the primordial *mode of “‘production’”* of humanity, that of ‘non-production’, i.e., of direct Appropriation of the products of Nature in their “Raw” forms, unimproved, for human consumption, by human labor, carried on by those primordial “bands” of predatory/foraging/scavenging hunter-gatherers. The «*arché*» for Dialectical Equation 7., $\mathbf{2}_{\mathbf{h}} \mathbf{R}_{\mathbf{h}, \mathbf{F}}$, connotes the primordial *human-society self-reproductive force* Resource of, or the primordial form of potential human-social “‘free energy’”/human-social “‘negentropy’”, harnessed/actualized by humanity, namely, that of the human community of those primordial “bands” of proto-human hunter-gatherers who carried on that “raw” Appropriation of the products of Nature.

The interweaving of dialectical, systems-progression ‘meta-models’ **1.** and **2.**, above, can be represented as follows --



-- which depicts progression **1.** progressing vertically, down its LHS [Left Hand Side], and progression **2.** progressing horizontally, with a distinct ‘sub-progression’ for each term of progression **1.** Progression **3.** can be brought in, by a further orthogonal progression-direction, normal to the page, intersecting the **2**nd sub-progression at its **2**nd term, w_{Q_L} :



The following links access extant **F.E.D.** texts which address the **2**nd, and the remaining **5** related, ‘dialectical meta-models’ --

2. [Meta-Systematic Dialectic of the Dialectical Arithmetics](#) Example (3), pages **3-01** to **3-12**
3. [Meta-Systematic Dialectic of Arithmetical/Algebraic Logics](#) pages **12** to **19**
4. [\[Psycho\]historical Dialectic of the Meta-Evolution of Arithmetics](#) pages **I-122** to **I-128**
5. [\[Psycho\]historical Dialectic of the Meta-Evolution of Human-Social Formation\(s\)](#) pages **III.A-01** to **III.A-29**
6. [\[Psycho\]historical Dialectic of the Social Relations of Production](#) pages **B-24** to **B-37**
7. [\[Psycho\]historical Dialectic of the Social Forces of Production](#) section **II**

Progressions **4.**, **5.**, **6.**, and **7.** tie-in to the three-fold interconnexion of the other three progressions at the level of that collective, human-phenomic cognitive “‘meta-evolution’”, a “‘meta-evolution’” which is driven by the development of the human-social forces of production/human-social relations of production praxis, which undergirds the **[psycho]historical dialectic** of ‘the meta-evolution of arithmetics’, as of so much else in the human phenome.

The “**N**atural Numbers”, in their advanced but still pre-“**a**ught numbers” representation, e.g., $\mathbf{N} \equiv \{ \mathbf{I}, \mathbf{II}, \mathbf{III}, \dots \}$, are associated with vestiges extending all of the way back into the earliest Terran human[oid] social **R**elations of production epoch, $\tau_R = \mathbf{0}$, with the predatory mode of self-reproduction of early hunting, gathering, and scavenging Paleolithic **b**ands, in the form of, e.g., notches carved in bone, to mark time by marking re-occurrences of periodically recurring events, such as full moons. The “**N**atural Numbers”, as the upper limit of human arithmetical «*mentalité*», thus correspond to the $2_{h_R} \mathbf{A}$ [**A**ppropriation of products of nature in their “raw” forms, not yet improved, for human use, by human labor] & early $2_{h_R} \mathbf{G}$ [**G**oods/obligatory **G**ifts production] “‘human-social **R**elations of **p**roduction’”/“‘**m**odes of [human-societal self-re-]**p**roduction’” [cf. Marx].

Something resembling what we would mean today by the phrase “the positive Rational Numbers”, i.e., the positive fractions, $Q_+ \equiv \{ (n_1/n_2) \mid n_1, n_2 \in \mathbf{N} \}$, may have emerged as early as, and in connexion with, the emergent Neolithic(\oplus) $^2_{\mathbf{h}}\mathbf{C}_{\mathbf{R}}$ [Commodity Barter] human-social Relation of [human-societal-re-]production, “‘The Commodity-Relation’” [cf. Marx], in response to pre-money/money-less exchange of Commodities, often forcing exchange-value-equating of non-Whole-number physical amounts of Commodities in barter exchange, among camp, then village, then chiefdom human-social formations.


Something resembling what we would mean today by the phrase “the positive Real numbers”, R_+ , i.e., the positive ‘[hexa-]decimalized’ fractions, used by the ‘later-ancient’ Babylonia astronomers, including potentially-infinitely-repeating and potentially-infinitely-non-repeating [hexa-]decimal, or anthypharesis, approximations, may have begun to emerge in association with late Neolithic(\oplus) agricultural production practices involving “land-measurement” or “earth-measurement” [“‘geo-metry’”], leading to the discovery of “incommensurable” geometrical magnitudes. This development belongs to the ancient Occidental classical period, in the Mediterranean venue, which increasingly featured ‘meta-chiefdom’ city-state, and “‘multi-city-state empire’” human-social formations, waxing region-wide and even into proto-global exchanges of Commodities, mediated by Monies, i.e., by $^2_{\mathbf{h}}\mathbf{M}_{\mathbf{R}}$, “The Money-Relation” [Marx], grasped as a social Relation of production. Eventually, even “antediluvian”, ‘protoic’, pre-industrial ‘pre-vestiges’ of “The Capital-Relation” [Marx] -- e.g., usurers’ «Kapital» and mercantile «Kapital» non-production forms, and chattel-slaves-worked, ‘latifundial’-plantation, capitalist agricultural production-forms of «Kapital» [forms of which re-appeared later, and persisted, in the colonial, confederation, pre-Civil-War federal, and Civil-War confederacy plantation-states of southern North America] -- emerged in the ancient Mediterranean world, all three representing historical «species» of «Kapital», $^2_{\mathbf{h}}\mathbf{K}_{\mathbf{R}}$, as a human-social Relation of production.



The formation, in advanced human-phenomic, collective cognition, of something resembling what we might describe today as the “non-negative Reals”, $R_{\geq 0}$, including the use [e.g., by Eudoxus [?] and Archimedes, e.g., in their “Method of Exhaustion” proto-integration algorithm] of informal/heuristic positive infinitesimals, verging on 0, and even of the use of 0 itself as a fully-operative number, constituting $R_{\geq 0}$, seems, in the Mediterranean locus, to have required the context of ancient classical multi-city-state empire social formations, at least, involving advanced Money-/proto-«Kapital»-based, proto-global commercial trading economies.

The acceptance, in the prevailing human-phenome, of a full “Real” numbers based -- R based -- conception of arithmetic, including of the negative “Reals”, belongs to the aftermath of the catastrophic contracted human-social reproduction of the last Western European “Dark Ages”, to the late-Medieval/Renaissance period’s resurrection of city-state republics, as well as of ‘protoic’ nation-state social formations, and of mercantile, banking, and, eventually, of large-scale manufacturing capitalism in that Occidental locus. That emergence of proto-industrial capitalism is epitomized by the use of oppositely-signed decimal numbers to present “debit” vs. “credit” Monetary-value & «Kapital»-value “‘quantifiers’” in the double-entry bookkeeping practices that also emerged, for the first time in human history, during that period.

The widespread acceptance and use of the Complex numbers, $C \equiv \{ Rr + Rri \}$ [which were first discovered/invented during Europe’s Renaissance rebirth from its last “Dark Ages”, by Rafael Bombelli, an engineer who worked in Italy, circa 1572 C.E./B.U.E], appears to belong to the zenith of the ascendant phase of the industrial capitalist epoch, within the sub-epoch of the prevalence of the nation-state social formation, and with the industrial development and application of “electro-chemical” productive forces, then of [sinusoidal alternating current] “electrical” productive forces [in whose formulae our i of the $^3_{\mathbf{H}}\mathbf{C}_{\#}$ “Standard Arithmetic” is traditionally notated as j instead, to help avert a potential confusion, given the traditional use of i to denote the electrical current variable], then of “electronic” productive forces, culminating, to-date, in sinusoidal EMR [ElectroMagnetic Radiation]-based, e.g., the radio wave/microwave technologies fundamental in production, transportation, & communication, epitomized by “phasor” sinusoidal dynamics, $e^{o\mathbf{i}} = \cos(\theta)r + \sin(\theta)i$.

These stages in the ‘meta-evolution of arithmetic’ correspond, of course, to ‘The [*Psycho*]*Historical*, diachronic *Dialectic* of Arithmetics’, not -- not directly at least -- to the essentially synchronic [*Meta*]-*Systematic Dialectical* presentation of

‘*The Gödelian Dialectic of the Standard Arithmetics*’ central to this essay. Nevertheless, the , the


 τ_R , and the
 
 τ_F cumula provide some of the historical, diachronic grounding which is still

ingredient -- however ‘complexly’ and implicitly so -- in the essentially synchronic view native to

$$\underbrace{{}^3\text{H}}_{\text{s}_\#}\left(\overset{\#}{\text{H}}\right)\text{ } \P .$$

Symbolic Economy, Semantic Density / Semantic Productivity, and Mnemonic Power.

The ‘Dialectical Meta-Equation’ that is our ‘meta-model’ of the systems of the “‘Standard Arithmetics’”--

$$\underbrace{{}^3\text{H}}_{\text{H}}\left(\overset{\#}{\text{H}}\right)_{\text{s}_{\#}}^{\uparrow} = \left({}^3\text{N}_{\text{H}_{\#}}\right)^{2\text{s}_{\#}\uparrow}$$

-- functions also as our *Encyclopedia Dialectica* definition of “‘Standard Arithmetic’”. As such, it is a dialectically “‘open-ended’” kind of definition. No final term, no ultimate ‘meta-meristem’, no forever-closing “‘culminant’”, is specified in this ‘meta-model’, by its ‘Dialectical Meta-Equation’. It remains a “‘*potentially infinite*’” [cf. Aristotle] sequence of equations, albeit one which is always, at any given moment of Terran human history, *actually* [meta]finite as to that part of its infinite *potential* which has been *actualized* so-far.

This ‘dialectical definition’ of “‘Standard Arithmetic’” is therefore not simply $\text{H}_{\text{H}_{\#}}^{\text{N}}$, or $\text{H}_{\text{H}_{\#}}^{\text{W}}$, or even $\text{H}_{\text{H}_{\#}}^{\text{C}}$, or $\text{H}_{\text{H}_{\#}}^{\text{H}}$, or $\text{H}_{\text{H}_{\#}}^{\text{O}}$. It is, on the contrary, the entire ‘meta-system-atic’, dialectical and cognitive movement from $\text{H}_{\text{H}_{\#}}^{\text{N}}$ to $\text{H}_{\text{H}_{\#}}^{\text{W}}$, from $\text{H}_{\text{H}_{\#}}^{\text{W}}$ to $\text{H}_{\text{H}_{\#}}^{\text{Z}}$, from $\text{H}_{\text{H}_{\#}}^{\text{Z}}$ to $\text{H}_{\text{H}_{\#}}^{\text{Q}}$, from $\text{H}_{\text{H}_{\#}}^{\text{Q}}$ to $\text{H}_{\text{H}_{\#}}^{\text{R}}$, from $\text{H}_{\text{H}_{\#}}^{\text{R}}$ to $\text{H}_{\text{H}_{\#}}^{\text{C}}$, from $\text{H}_{\text{H}_{\#}}^{\text{C}}$ to $\text{H}_{\text{H}_{\#}}^{\text{H}}$, from $\text{H}_{\text{H}_{\#}}^{\text{H}}$ to $\text{H}_{\text{H}_{\#}}^{\text{O}}$, *and beyond*, summarized by/“contained in”, that ‘Dialectical Meta-Equation’ --

$$\begin{array}{cccccccccccc} \text{H}_{\text{H}_{\#}}^{\text{N}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{W}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{Z}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{Q}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{R}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{C}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{H}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{O}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{K}} & \rightarrow & \text{H}_{\text{H}_{\#}}^{\text{G}} & \rightarrow & \dots \\ \text{s: } 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8 & \rightarrow & 9 & \rightarrow & 10 & \rightarrow & \dots \end{array}$$

-- both *actually* [e.g., to $\text{H}_{\text{H}_{\#}}^{\text{K}}$ and to $\text{H}_{\text{H}_{\#}}^{\text{G}}$], and *potentially* [to arithmetics beyond those that have been *actualized* -- codified or axiomatized -- by Terran humanity to-date], given the “*incompleteness or inexhaustibility*” of mathematics in general, and of arithmetics in particular, established by Gödel.

However, being confronted by the potential infinity of symbols that are required by such encyclopedic dialectical-equational definitions, we must grapple with issues of the ease and compactness of their ‘representability’ via our ‘dialectical arithmetics’, and via their ‘dialectical algebras’. Doing so, we find that our situation is, indeed, quite favorable in that regard. If we strip the ‘Dialectical Meta-Equation meta-model’ that forms the core of this essay down to its bare essentials, stripping off all of the helpful but inessential taxonomical locator epithets, or ‘dialectical diacritical marks’, then our most condensed concentration of the meaning of this entire essay requires just four symbols, or just four “‘symbolic elements’”, namely, the elements ‘ ’, ‘**N**’, ‘**2**’, and ‘**6**’, arrayed as follows --

$$\text{N}^{26}$$

-- such that the **4** symbolic-elements above, so arranged, can replace, e.g., the entire **64**-term, ≈ 641 symbolic-element expression that concludes the core section of this essay. They can do so in this sense: the entire **64**-term series can be re-constituted and recovered, from the **4** symbol, ‘semantically concentrated’ version, simply by repeatedly applying **3** simple rules -- i.e., just **3** of the **9** core axioms of the $\text{H}_{\text{H}_{\#}}^{\text{Q}}$ system of dialectical arithmetic, as given herein within section **B.1.** -- namely, Axioms **§7**, **§8**, and **§9** --

(§7) $[\forall n \in \text{N}][\text{Q}_{\text{H}_n} \boxplus \text{Q}_{\text{H}_n} = \text{Q}_{\text{H}_n}]$ [the axiom of *idempotent addition* / of *ontological category* [ontological qualifier] *uniqueness*].

(§8) $[\forall i, j, k \in \text{N}][[j \succ k] \Rightarrow [\text{Q}_{\text{H}_j} \boxplus \text{Q}_{\text{H}_k} \nrightarrow \text{Q}_{\text{H}_i}]]$ [the axiom of the *irreducibility* of ontological *qualitative differences*].

(§9) $[\forall j, k \in \text{N}][[\text{Q}_{\text{H}_j} \boxtimes \text{Q}_{\text{H}_k} = \text{Q}_{\text{H}_k} \boxplus \text{Q}_{\text{H}_{k+j}}]]$ [the axiom of the *double-«aufheben»* evolve product rule for ontological multiplication].

-- and by one or more applications of the ‘Organonic Algebraic Method’ to “‘re-solve-for’” any once-known but no-longer-known/-remembered terms, when the meanings of some of them are forgotten subsequent to reading this essay.

If we take the “‘replacement rate’” -- the percent-ratio of the count of the number of *terms* replaced to that of the *symbolic elements* so replacing -- as a crude metric for the degree of ‘*semantic compression*’, or of ‘*knowledge-representation-condensation*’, achieved, then the ‘semantic density’ improvements that we are achieving by using the stripped down, dialectical, Dyadic Seldon Function formulations, are impressive, viz. --

- out to $\overset{3}{\underset{\#}{H}}\overset{5}{\textcolor{blue}{R}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $32/4 = 8 = 800\%$ ‘*semantic condensation rate*’;
- out to $\overset{3}{\underset{\#}{H}}\overset{6}{\textcolor{blue}{C}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $64/4 = 16 = 1,600\%$ ‘*semantic condensation rate*’;
- out to $\overset{3}{\underset{\#}{H}}\overset{7}{\textcolor{violet}{H}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $128/4 = 32 = 3,200\%$ ‘*semantic condensation rate*’;
- out to $\overset{3}{\underset{\#}{H}}\overset{8}{\textcolor{brown}{O}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $256/4 = 64 = 6,400\%$ ‘*semantic condensation rate*’;
- out to $\overset{3}{\underset{\#}{H}}\overset{9}{\textcolor{black}{K}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $512/4 = 128 = 12,800\%$ ‘*semantic condensation rate*’;
- out to $\overset{3}{\underset{\#}{H}}\overset{10}{\textcolor{blue}{G}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $1,024/5 \approx 204 = 20,400\%$ ‘*semantic condensation rate*’;
- out to $\overset{3}{\underset{\#}{H}}\overset{11}{\textcolor{black}{X}}$ and its «*aporia*», $\overset{2}{\textcolor{red}{N}}$: $2,048/5 \approx 410 = 41,000\%$ ‘*semantic condensation rate*’;

Using the ‘minimalized’ Seldon Function format -- $\overset{s}{\underset{v}{\alpha}}$ -- the systematic(s) core of a discourse: of a whole lecture, or of a whole text -- paper, essay, book, multi-«*buch*»/multi-volume treatise, etc. ... -- can be mnemonically summarized, using as few as four *symbolic elements*, in an expression which, with the application of three rules, & of the ‘organonic method’, if needed, can, at will, be quickly reconstituted into a series/sum/cumulum of tens, or hundreds, or thousands,... of *terms*, capturing, in systematically-ordered detail, the gist of the content of that discourse.

That ‘minimalized’ Seldon Function format can formulate condensed, ‘re-implicitized’, ‘connotationally curtailed’, or ‘darkened’, “‘black [w]holes’” of information, from which “‘white [w]holes’” of outpouring “‘[w]holistic’”/mnemonic re-elaboration and reconstitution of that information are ever ready to be ‘re-unfolded’, to be ‘re-unfurled’, to be ‘rotely’ ‘re-burged’, by those who know the 3 axiomatic rules [and the ‘organonic method’].

The mere assertion of a category, within a specific, interpreted progression/sum, or ‘[ac]cumulum’, of categories, is not, in itself, the delineation and articulation, or ‘explicitization’, of the detailed content -- of the progression/sum/cumulum of sub-categories and of sub-sub-categories... which are implicit in that category when it is asserted as an unarticulated, undelineated, undivided, univocal whole. But the assertion of that undivided category does serve as a collective name for, and as a reminder of -- an intimation of -- the content of that category in its more fully articulated detail, as experienced/-conducted in the past, and as still ‘rememberable’, to some degree, by the user, presently.

Of course, in the last analysis, the ‘categorogram’ or ‘category ideogram’ symbols, that constitute these dialectical progression expressions, are “‘intensional symbols”, not “‘extensional symbols”. Each is a ‘connotogram’, not an explicit list of symbols in 1-1 correspondence with “every last” element of meaning of the [ideo-]ontological category that it represents. The meanings of those ‘categorograms’ are not “all there in the symbols”, and such *they never can be*. What each is, is a ‘mnemonic trigger’, an ‘associational catalyst’, to remind the user of, and to help [re-]evoke in the user, the rich totality of ‘implicit semanticities’ that these “‘intensional” symbols intend.

The richer the web of associations, of previously constructed and ‘re-member-éd’ knowledge -- of remembered experience in general -- that the user brings to those symbols, evoked in the user’s past, and retained in mind, i.e., in the user’s ‘meme-ory’, ever since, the richer, then, the totality of meanings that these ‘*semantically densified*’ and ‘*semantically concentrated*’ symbols “‘hold’” *for that user*, and the better the odds for that user to evoke that richness *in and for others*.

Endnotes.

¹ Caveat: I do not claim to have reproduced, in the ‘meta-model’ herein presented, ‘The Gödelian Dialectic’ in the sense of having discovered an assignment of the “**N**atural Numbers” to the elementary constant signs, numerical variables, sentential variables, and predicate variables of the predicate calculus such that --

- α.** The “Gödel Formula” for the $\overset{3}{H}\overset{\#}{N}$ axioms-system “deformalizes” to an $\overset{3}{H}\overset{\#}{N}$ -undecidable, but true, proposition asserting that a diophantine equation of form $n - n = x_\alpha, n \in \mathbf{N}$, is unsolvable in $\overset{3}{H}\overset{\#}{N}$, & such that;
- β.** The “Gödel Formula” for the $\overset{3}{H}\overset{\#}{W}$ axioms-system “deformalizes” to a $\overset{3}{H}\overset{\#}{W}$ -undecidable, but true, proposition stating that a diophantine equation $w + x_\beta = 0; x_\beta, w \neq 0, w \in \mathbf{W}$, is unsolvable in $\overset{3}{H}\overset{\#}{W}$, & such that;
- γ.** The “Gödel Formula” for the $\overset{3}{H}\overset{\#}{Z}$ axioms-system “deformalizes” to a $\overset{3}{H}\overset{\#}{Z}$ -undecidable, but true, proposition asserting that an inequation of form $|x_\gamma \times z| < |z|, |x_\gamma| > 0, z \in \mathbf{Z}$, is unsolvable in $\overset{3}{H}\overset{\#}{Z}$, & such that;
- δ.** The “Gödel Formula” for the $\overset{3}{H}\overset{\#}{Q}$ axioms-system “deformalizes” to a $\overset{3}{H}\overset{\#}{Q}$ -undecidable, but true, proposition asserting that a diophantine equation of form $x_\delta^2 = p, 0 < p \in \mathbf{Q}$, and with **p** denoting a [positive] [rational] p prime number, is unsolvable in $\overset{3}{H}\overset{\#}{Q}$, i.e., that such $x_\delta \notin \mathbf{Q}$, & such that;
- ε.** The “Gödel Formula” for the $\overset{3}{H}\overset{\#}{R}$ axioms-system “deformalizes” to a $\overset{3}{H}\overset{\#}{R}$ -undecidable, but true, assertion that an equation of form $x_\epsilon^2 + r = 0, 0 < r \in \mathbf{R}$, is unsolvable in $\overset{3}{H}\overset{\#}{R}$, i.e., that such $x_\epsilon \notin \mathbf{R}$, & such that;
- ζ.** The “Gödel Formula” for the $\overset{3}{H}\overset{\#}{C}$ axioms-system “deformalizes” to a $\overset{3}{H}\overset{\#}{C}$ -undecidable, but true, proposition asserting that a diophantine equation $+x_\zeta y_\zeta = -y_\zeta x_\zeta | x_\zeta, y_\zeta \neq 0$, is $\overset{3}{H}\overset{\#}{C}$ -unsolvable, i.e., that such $x_\zeta, y_\zeta \notin \mathbf{C}; \dots$

What I do claim is that **F.E.D.** has selected, for ‘pedagogical-strategic’ reasons, “unsolvable” [in]equations of the forms given above, in the order given above, in the spirit of the Gödel First Incompleteness Theorem. That is, those [in]equations are suggestive of the logical-incompleteness-establishing Gödel Formulae, as “‘unprovable theorems’”, “deformalizing” to propositions asserting that certain diophantine equations are unsolvable within the arithmetical axioms-system in which they arise as ‘well-formed [in]equations’, and portending expanded arithmetical axioms-systems, featuring new kinds of numbers, in which the “‘unprovable theorems’” become provable, in which the “deformalization” propositions become deductively demonstrable, and by which new kinds of numbers their “unsolvable” diophantine [in]equations become solvable.

² A crucial resource for becoming [re-]sensitized to the profound psychohistorical/memetic chasm which separates the ancient-Mediterranean-plus consensus conception of “‘number’” as «arithmos», and the same-name-translated, but semantically vastly disparate concept of “number” endemic to Occidental modernity, is that magisterial «opus» by Jacob Klein, entitled Greek Mathematical Thought and the Origin of Algebra, Dover, [NY: 1968]. A key [psycho]historical dimension of the **F.E.D.** Immanent Critique of Arithmetic [ICOA] is the Seldonian discovery, and the subsequent **F.E.D.** recurrent re-evocation, of a ‘dialectical synthesis-sum’ that breaks through the still-dominant human-Phenomic «aporia» between the Ancient conception of ‘qualo-quantitative’ numbers [«arithmoi»], including both the “logistical” numbers [the «arithmoi monadikoi»], & the Platonian conception of “unaddable” [«asumblatoi»], ‘purely-qualitative’ [dialectical] numbers’ [the «arithmoi eidetikoi»], $\overset{3}{H}\overset{\#}{A}_P$, as dialectical thesis, &/versus its ‘dialectical contra-thesis’ of the Modern conception of “‘symbolical, purely-quantitative numbers’”, $\overset{3}{H}\overset{\#}{M}_P$, as per the following Triadic Seldon Function --

$$\overset{3}{H}\overset{\#}{I}\overset{P}{1} = \left(\overset{3}{H}\overset{\#}{A}_P \right)^{3^1} = \left(\left(\overset{3}{H}\overset{\#}{A}_P \right) \oplus \left(\overset{3}{H}\overset{\#}{M}_P \right) \right) \oplus \left(\overset{3}{H}\overset{\#}{MA}^P \right), \text{ seeding Seldon's [psycho]historic[al] breakthrough.}$$

³ For some ‘theorem-etical’ considerations on the axiomatics of nQ, wQ, and zQ, plus much else besides, see the recent **E.D. Briefs** -- **Briefs** #5, #6, & #7 -- by a www.dialectics.org guest author, code-name “J2Y”: Cumulation-Spaces.